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## Abstract

This paper develops a piecewise linear toolkit for optimal policy analysis of linear rational expectations models, subject to occasionally binding constraints on (multiple) policy instruments and other variables. Optimal policy minimises a quadratic loss function under either commitment or discretion. The toolkit accounts for the presence of 'anticipated disturbances' to the model equations, allowing optimal policy analysis around scenarios or forecasts that are not produced using the model itself (for example, judgement-based forecasts such as those often produced by central banks). The flexibility and applicability of the toolkit to very large models is demonstrated in a variety of applications, including optimal policy experiments using a version of the Federal Reserve Board's FRB/US model.



**Key words:** Optimal policy, commitment, discretion, occasionally binding constraints.

**JEL classification:** C61, C63, E61.



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# 1 Introduction

Recent macroeconomic and policy developments have led to an increased study of occasionally binding constraints. Following the global financial crisis of 2007–2008, monetary policymakers in many economies cut their short-term policy rates to their effective lower bounds. The development of macro-prudential frameworks has introduced policy instruments, such as capital requirements, that may bind occasionally.

These events have prompted the development of methods to incorporate occasionally binding constraints in policy-relevant macroeconomic models. The typical size of these models prohibits the use of global solution methods, which properly account for how the *risk* that constraints bind in the future will affect the actions of forward-looking decision makers today. As a result, many techniques focus on the inclusion of occasionally binding constraints in an otherwise linear-quadratic framework.

This paper contributes to that research effort by consolidating, combining and extending existing approaches into a toolkit that is capable of analyzing a wide range of policy-relevant scenarios.

The tools developed in this paper apply to linear rational expectations models, of the type routinely used in policy institutions. The tools focus on cases in which policy is conducted optimally, to minimize a quadratic loss function, under either commitment or discretion (optimal time-consistent policy). The paper presents methods for cases in which both policy instruments and non-policy variables may be subject to occasionally binding constraints.

The methods build on the 'Model Analysis & Projection System' (MAPS) toolkit (Burgess et al., 2013). That toolkit incorporates the Anderson and Moore (1985) solution algorithm for linear rational expectations models, which allows the solution to be expressed in terms of the expected future paths of exogenous variables. In this paper, the future paths of exogenous variables are treated as 'anticipated disturbances' to the model equations. Following Svensson and Tetlow (2005), these anticipated disturbances can be interpreted as capturing judgment and other 'off-model' information that informs the scenarios and forecasts produced by many policy institutions. Incorporating these disturbances therefore makes it possible to conduct optimal policy analysis with occasionally binding constraints for projections and scenarios produced using judgment.

The toolkit developed in this paper has several advantages over existing methods (discussed below). By providing an integrated framework, the toolkit facilitates the comparison of optimal policy under alternative assumptions about the policymaker's ability to commit to future actions. The toolkit easily supports multiple policy instruments and multiple constraints, which is likely to be particularly useful in light of the increasing use of additional unconventional monetary policy tools. Moreover, where possible, the toolkit is designed to ensure that incorporating additional constraints does not materially affect the 'scale' of the solution problem. This makes it particularly relevant for applications with large-scale models typically used by policy institutions.[1]

The methods in this paper contribute to a strand of research focused on the incorporation of occasionally binding constraints in an otherwise linear-quadratic framework. Indeed, the toolkit builds heavily on some key contributions to the literature.

The tools for analyzing optimal commitment problems combine the optimal policy solution set out in Dennis (2007) with the method for imposing occasionally binding constraints developed by Holden and Paetz (2012) and Holden (2019). While the Holden and Paetz (2012) method provides a powerful method for incorporating the effects of instrument bounds under commitment, it cannot in general be directly applied to optimal discretionary policy. The algorithms for optimal discretion therefore combine the approaches of Dennis (2007) and Brendon et al. (2010), extended to incorporate the presence of 'anticipated disturbances'. Indeed, a common thread connecting all of the methods presented in this paper is the importance of these disturbances in permitting optimal policy analysis of non-model-based

[1] The toolkit, together with replication code for the examples in the paper, is available from the authors on request.

forecasts and scenarios, building on the insights of Svensson and Tetlow (2005).

A popular and powerful approach for incorporating occasionally binding constraints is the 'OccBin' toolkit developed by Guerrieri and Iacoviello (2015). In principle, this approach can be applied to optimal commitment problems with constraints on the policy instruments, since the solution concept is identical to that studied by Holden and Paetz (2012). However, such an implementation would have several practical disadvantages. In particular, it would require derivation and specification of the first order conditions for optimal policy within the model. It would also be less easily scalable to the case of multiple constraints and would not permit the application of anticipated disturbances, thus limiting the range of feasible applications.

Other methods that can be used to analyze occasionally binding constraints are similarly specialized and therefore not directly applicable to the range of applications that can be studied with the toolkit developed in this paper. For example, approaches that embed anticipated structural change have been used to incorporate the lower bound on the policy instrument (see, for example, Cagliarini and Kulish, 2013; Kulish and Pagan, 2017; Kulish et al., 2017). However, these methods do not do not allow for policy to be set optimally. Similar issues apply to methods that incorporate Markov switching methods to impose occasionally binding constraints (Bianchi and Melosi, 2017; Chen, 2017; Benigno et al., 2020).

This paper is also part of an enormous literature on optimal monetary policy in linear quadratic rational expectations models. While many contributions consider more sophisticated and challenging policy environments, such as stochastic structural change (Blake and Zampolli, 2011; Svensson and Williams, 2008) and leader/follower relationships between different policymakers (Chen et al., 2020), they do not incorporate occasionally binding constraints on the policy instruments. Implementing occasionally binding constraints within those frameworks would be a worthwhile avenue for future research.

This paper is most closely related to de Groot, Mazelis, Motto, and Ristiniemi (2021), who have, in independent work, developed a toolkit (called 'COPP') for optimal policy analysis. Similarly to the methods presented below, COPP can analyze optimal time-consistent and commitment policies using multiple policy instruments that are subject to occasionally binding constraints. It is also designed for optimal policy analysis around a scenario or projection that is not generated by the model itself. An important difference is that the COPP toolkit does not require a fully specified structural model, merely a baseline path for the variables in the policymaker's loss function and impulse responses of those variables to changes in the paths of the policy instruments. While similar in several respects, the methods described below employ a different definition of time-consistent optimal policy and are also designed to handle cases with occasionally binding constraints on non-instrument variables.

Piecewise linear solution methods (such as those presented in this paper and discussed above) represent an approximation to the true effects of occasionally binding constraints in rational expectations models. These methods do not account for the fact that agents' expectations account for the risks of encountering the occasionally binding constraints in future periods. These effects on expectations can have substantial effects on current decisions and hence the true model solution. Incorporating these effects requires sophisticated solution methods, typically employing global solution techniques. The pace of progress in this area is impressive.[2] However, at present, the methods are typically applicable only to relatively small models. Moreover, piecewise linear approximations may be sufficiently accurate in some important cases, as demonstrated by Guerrieri and Iacoviello (2015).

After setting out key assumptions and notation in Section 2, the rest of the paper paper proceeds incrementally, using simple examples to demonstrate the new functionality introduced in each section.

Section 3 presents the method for optimal commitment policy, subject to occasionally binding constraints on the policy instruments. Section 4 describes the method for optimal discretionary policy, when there are anticipated disturbances to the model equations. This serves as a stepping stone to Section

[2]Important recent contributions to this field include Brumm and Scheidegger (2017), Druedahl and Jørgensen (2017) and Maliar et al. (2019).

5, which considers optimal discretionary policy subject to occasionally binding constraints on the policy instruments. Section 6 presents methods for handling cases in which both policy instruments and other variables are subject to occasionally binding constraints.

Section 7 complements the examples from previous sections by utilizing FRB/US, a large-scale model used by the Federal Reserve Board of Governors, to demonstrate the applicability of the toolkit to policy-relevant questions. Section 8 concludes.

## 2 Preliminaries

The methods in this paper build on the 'Model Analysis & Projection System' (MAPS) toolkit developed at the Bank of England (Burgess et al., 2013). That toolkit is designed to operate with discrete time, infinite horizon linear(ized) rational expectations models. The discrete 'period' or 'date' is indexed by $t = 1, \ldots, \infty$ and the model is written in the following form:

$$H^F \mathbb{E}_t x_{t+1} + H^C x_t + H^B x_{t-1} = \Psi z_t \tag{1}$$

where $x_t$ is a $n_x \times 1$ vector of endogenous variables, $z_t$ is a $n_z \times 1$ vector of exogenous disturbances, $\mathbb{E}_t$ represents the mathematical expectation conditional on period-$t$ information. The $n_x \times n_x$ matrices $H^F, H^C, H^B$ and the $n_x \times n_z$ matrix $\Psi$ are coefficient matrices.

A wide range of models can be cast in this form. A prominent example is the class of so-called Dynamic Stochastic General Equilibrium (DSGE) models, typically consisting of a set of first order conditions to optimization problems and the constraints on those problems. Taking a log-linear approximation to these equations around a suitable approximation point generates a set of linear equations of the form (1). In these cases, the entries in the coefficient matrices $(H^F, H^C, H^B, \Psi)$ will be functions of so-called 'deep' parameters describing preferences and technology.

The form in (1) omits constants explicitly, which is typically appropriate for log-linearized DSGE models. However, constants can be incorporated into a model as an element of $x$, with appropriate adjustments to the conditions governing existence and uniqueness of a rational expectations equilibrium.[3] Similarly, higher order lags and leads of endogenous variables can be incorporated by introducing appropriate identities to define auxiliary variables.

The rational expectations solution to the model (1) is given by:

$$x_t = Bx_{t-1} + \mathbb{E}_t \sum_{i=0}^{\infty} F^i \Phi z_{t+i} \tag{2}$$

where the solution matrix $B$ can be computed using the 'AIM' algorithm Anderson and Moore (1985) and:

$$F = -\left(H^C + H^F B\right)^{-1} H^F \tag{3}$$

$$\Phi = \left(H^C + H^F B\right)^{-1} \Psi \tag{4}$$

The rational expectations solution (2) (and variants of it) form the basis of much of the analysis in this paper. The solution includes the expected values of future disturbances, $\mathbb{E}_t z_{t+i}$.[4]

[3] As in Hansen and Sargent (2013), an element in $x$ that captures the constants (say, $\iota$) can be defined using the equation $\iota_t = \iota_{t-1}$. Ensuring that this variable has the initial condition $\iota_{-1} = 1$ allows constants to be included by loading on $\iota$ with the appropriate coefficient. This approach implies that there will be an exact unit root in the model which should be accounted for in the standard 'root counting' tests (eg Blanchard and Kahn (1980)) for existence and uniqueness of a rational expectations equilibrium.

[4] Under the common assumption that shocks are serially uncorrelated, mean zero random variables, then $\mathbb{E}_t z_{t+i} = 0, \forall i$ and the rational expectations solution can be written as $x_t = Bx_{t-1} + \Phi z_t$.

The representation of the solution (2) has three important strengths. First, it provides a route through which expectations of future constraints may be incorporated – as explained in detail below. Second, it permits the analysis of 'news shocks' without the need to expand the state vector of the the model to incorporate them.[5] Finally, Svensson and Tetlow (2005) demonstrate how anticipated disturbances can be used to ensure that a structural model (such as that in equation (1)) reproduces a forecast generated from other sources. In particular, the disturbances $z$ can be interpreted as judgmental information that lies beyond the scope of the model.

Indeed, this final point represents a key contribution of the methods in this paper. Specifically, the methods support the interpretation in Svensson and Tetlow (2005) that: "the central bank's judgment will be represented as the central bank's projections of the future deviations [that is, $\{z_{t+i}\}_{i=0}^{\infty}$]. This allows us to incorporate the fact that a considerable amount of judgment is always applied to assumptions and projections."

So, as in Svensson and Tetlow (2005), the methods presented in this paper, allow a structural model to be used to conduct optimal policy analysis around a non-model-based forecast. Many central banks and policy institutions use a broad range of information to produce their primary forecasts and large-scale structural models of the form (1) for policy analysis. The ability to combine model-based optimal policy analysis with judgment-based forecasts therefore represents a key strength of the methods presented below. Section 7 discusses this point in more detail and provides an example.

The terms $\mathbb{E}_t z_{t+i}$ in (2) can also be used to encode information about future constraints on variables in the model, including policy instruments. For example, Laséen and Svensson (2011) use $\mathbb{E}_t z_{t+i}$ to impose expected future paths for the monetary policy instrument in DSGE models in which monetary policy follows a simple rule. That is achieved by using (2) to compute the sequence of future shocks to the monetary policy rule that deliver a desired path for the short-term interest rate.[6] This approach has been applied to study the responses of DSGE models to 'forward guidance' about the path of the monetary policy instrument.[7]

The methods in this paper focus on cases in which policy is conducted optimally. The starting point is a version of the model in which the policy rules describing the behavior of the policy instruments are removed from the model (1). Doing so gives the following representation:

$$\widetilde{H}^F_{\widetilde{x}} \mathbb{E}_t \widetilde{x}_{t+1} + \widetilde{H}^C_{\widetilde{x}} \widetilde{x}_t + \widetilde{H}^B_{\widetilde{x}} \widetilde{x}_{t-1} + \widetilde{H}^F_r \mathbb{E}_t r_{t+1} + \widetilde{H}^C_r r_t = \widetilde{\Psi}_{\widetilde{z}} \widetilde{z}_t \tag{5}$$

where $\widetilde{x}$ denotes the $n_{\widetilde{x}} \times 1$ vector of non-policy endogenous variables, $r$ is the $n_r \times 1$ vector of policy instruments and $\widetilde{z}$ denotes the $n_{\widetilde{z}} \times 1$ vector of non-policy shocks.

To move from (1) to (5), the $n_x (= n_{\widetilde{x}} + n_r)$ variables, $x$, are partitioned into policy instruments, $r$ and non-policy variables $\widetilde{x}$. The $n_r$ equations that describe the behavior of the policy instruments are removed, leaving a system of $n_{\widetilde{x}}$ equations. The matrices $\widetilde{H}^F_{\widetilde{x}}, \widetilde{H}^C_{\widetilde{x}}, \widetilde{H}^B_{\widetilde{x}}$ are $n_{\widetilde{x}} \times n_{\widetilde{x}}$ coefficient matrices formed by extracting the relevant rows and columns from $H^F, H^C, H^B$. The $n_{\widetilde{x}} \times n_r$ coefficient matrices $\widetilde{H}^F_r$ and $\widetilde{H}^C_r$ are constructed analogously. The $n_{\widetilde{x}} \times n_{\widetilde{z}}$ matrix $\widetilde{\Psi}_{\widetilde{z}}$ is found by removing the rows of $\Psi$ corresponding to the policy equations and the columns corresponding to any policy shocks (that appear solely in the policy equations).

Without loss of generality, lags of the policy instruments, $r$, are excluded from (5).[8] This ensures

[5]There is a rich literature examining news shocks in models of the type studied in this paper. Beaudry and Portier (2014) provide a broad overview and Gambetti et al. (2019) contribute to (and review) the more recent literature with a focus on monetary policy.

[6]Burgess et al. (2013, Appendix C) presents a general purpose 'inversion algorithm' to compute the required sequences of anticipated disturbances $\mathbb{E}_t z_{t+i}$ required to generate a desired forecast for the endogenous variables, $\mathbb{E}_t x_{t+i}$.

[7]See, for example: Hirose and Kurozumi (2011); Milani and Treadwell (2012); Harrison (2015); Haberis et al. (2019).

[8]This is not a restrictive assumption since any model that does contain lags of the instrument(s) can be

that the instruments are not state variables in the system, thereby simplifying the form that the solution takes.

The policymaker chooses the instruments $r$ to minimize a quadratic loss function given by:

$$\mathcal{L}_t \equiv \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ (\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + (r_{t+i})' Q (r_{t+i}) \right\} \tag{6}$$

where $0 < \beta < 1$ is the policymaker's discount factor and $W$ and $Q$ are $n_{\widetilde{x}} \times n_{\widetilde{x}}$ and $n_r \times n_r$ positive semi-definite weighting matrices.

As in Dennis (2007), cases in which loss function minimization is implemented under commitment and discretion are considered.

Under commitment, the policymaker formulates a time-invariant policy plan that accounts for the entire sequence of constraints on the minimization problem. The Lagrange multipliers on these constraints appear in the first order conditions of the optimal policy problem and provide a link through which future policy is influenced by current multipliers. A corollary is that promises of future policy actions can affect outcomes today.

Under discretion, the precise concept of equilibrium is 'Markov-perfect Stackelberg-Nash'. Such equilibria arise from the following assumptions. At the start of each period, the policymaker observes the state of the economy and then sets the instruments optimally. Having observed the policymaker's optimal instrument setting, the private sector makes their decision, given rational expectations about the future state of the economy. Both the policymaker and the private sector take the future optimal behavior of the policymaker as given when making their decisions.[9]

# 3 Optimal commitment with instrument bounds

The method to implement an optimal commitment solution with bounds on the policy instrument combines the analysis of Dennis (2007) and Holden and Paetz (2012).

## 3.1 Method

The policymaker minimizes the discounted sum of current and future losses subject to the structural equations of the economy and bound constraints on the policy instruments. The optimal commitment solution can be written as a Lagrangean, in the same way as Dennis (2007) for the unconstrained problem and Harrison (2012) for the case in which the instruments are constrained:

$$\widetilde{\mathcal{L}}_t = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left[ \begin{array}{c} (\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + (r_{t+i})' Q (r_{t+i}) \\ -2\lambda'_{t+i} \left( \widetilde{H}^F_{\widetilde{x}} \mathbb{E}_t \widetilde{x}_{t+1} + \widetilde{H}^C_{\widetilde{x}} \widetilde{x}_t + \widetilde{H}^B_{\widetilde{x}} \widetilde{x}_{t-1} + \widetilde{H}^F_r \mathbb{E}_t r_{t+1} + \widetilde{H}^C_r r_t - \widetilde{\Psi}_{\widetilde{z}} \widetilde{z}_t \right) \\ -2\mu'_{t+i} (S r_{t+i} - b) \end{array} \right]$$

where $\lambda_{t+i}$ and $\mu_{t+i}$ are the Lagrange multipliers on the structural equations and the instrument bound(s) respectively. The instrument bounds are encoded by $S$, a coefficient matrix, and $b$, an $n_\mu \times 1$ vector of constants.

rewritten (by introducing appropriate identities if necessary) so that no instrument lags appear.

[9] The equilibria are Markovian because they are characterized by feedback rules from the state of the economy inherited from the previous period (and not the entire history). They are sub-game perfect, Stackelberg-Nash because they can be derived as Nash equilibria in a game between current and future policymakers in which the current policymaker acts as Stackelberg leader and in which the strategy pursued by the policymaker is optimal regardless of time *and* regardless of the state of the economy.

The first order conditions of the problem can be written as:

$$0 = Qr_t - \left(\widetilde{H}_r^C\right)' \lambda_t - \beta^{-1} \left(\widetilde{H}_r^F\right)' \lambda_{t-1} - S'\mu_t \tag{7}$$

$$0 = W\tilde{x}_t - \left(\widetilde{H}_{\tilde{x}}^C\right)' \lambda_t - \beta^{-1} \left(\widetilde{H}_{\tilde{x}}^F\right)' \lambda_{t-1} - \beta \left(\widetilde{H}_{\tilde{x}}^B\right)' \mathbb{E}_t\lambda_{t+1} \tag{8}$$

$$0 = \widetilde{H}_{\tilde{x}}^F \mathbb{E}_t\tilde{x}_{t+1} + \widetilde{H}_{\tilde{x}}^C \tilde{x}_t + \widetilde{H}_{\tilde{x}}^B \tilde{x}_{t-1} + \widetilde{H}_r^F \mathbb{E}_t r_{t+1} + \widetilde{H}_r^C r_t - \widetilde{\Psi}_{\tilde{z}}\tilde{z}_t \tag{9}$$

$$0 = \mu_t' \left(Sr_t - b\right) \tag{10}$$

for $t > 1$ (the first order conditions for $t = 0$ are considered below).

Equation (10) is the Kuhn-Tucker condition on the instrument inequality constraint. It pins down the value of the multiplier $\mu$ depending on whether the optimality condition (7) can be achieved without the instrument(s) violating the bound(s).

The first order conditions that apply in period $t = 1$ depend on the assumptions about the interpretation of the policy problem solved in that period. Taking a 'timeless perspective' to the optimal policy problem implies that the policymaker behaves as if the first period of optimization was in the distant past (Woodford, 1999). From a practical perspective, this implies that the first order conditions (7)–(10) apply in period $t$, for some exogenously specified values of the multipliers, $\lambda_0$. An alternative approach is to interpret the optimization at $t = 1$ as entirely independent of any previous commitments (so that $\lambda_0 = 0$).

There has been much debate about the appropriateness of the timeless perspective approach to optimal policy analysis.[10] It is beyond the scope of this paper to contribute to that debate. For completeness, solutions are presented for both timeless perspective and fully optimal cases.

### 3.1.1 Timeless perspective

To proceed, $\mu$ is treated as if it is exogenous. As will be demonstrated below, this allows $\mu$ to be used as a 'shadow shock' to impose the instrument bounds on the model solution in a way that is entirely consistent with the endogeneity of $\mu$.

This assumption allows the system (7)–(9) to be stacked together as:

$$0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta\left(\widetilde{H}_{\tilde{x}}^B\right)' \\ \widetilde{H}_{\tilde{x}}^F & \widetilde{H}_r^F & 0 \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{x}_{t+1} \\ r_{t+1} \\ \lambda_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & Q & -\left(\widetilde{H}_r^C\right)' \\ W & 0 & -\left(\widetilde{H}_{\tilde{x}}^C\right)' \\ \widetilde{H}_{\tilde{x}}^C & \widetilde{H}_r^C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_t \\ r_t \\ \lambda_t \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & -\beta^{-1}\left(\widetilde{H}_r^F\right)' \\ 0 & 0 & -\beta^{-1}\left(\widetilde{H}_{\tilde{x}}^F\right)' \\ \widetilde{H}_{\tilde{x}}^B & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ r_{t-1} \\ \lambda_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & -S' \\ 0 & 0 \\ -\tilde{\Psi}_{\tilde{z}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{z}_t \\ \mu_t \end{bmatrix} \tag{11}$$

This system can be written as a structural model in the augmented vector of endogenous variables:

$$y_t \equiv \begin{bmatrix} \tilde{x}_t \\ r_t \\ \lambda_t \end{bmatrix} \tag{12}$$

giving:

$$H_y^F \mathbb{E}_t y_{t+1} + H_y^C y_t + H_y^B y_{t-1} = \Psi_y \tilde{z}_t + \Psi_\mu \mu_t \tag{13}$$

[10] See, for example: Blake and Kirsanova (2004); Dennis (2010); Jensen and McCallum (2010).

where:

$$H_y^F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta\left(\widetilde{H}_{\widetilde{x}}^B\right)' \\ \widetilde{H}_{\widetilde{x}}^F & \widetilde{H}_r^F & 0 \end{bmatrix}$$

$$H_y^C = \begin{bmatrix} 0 & Q & -\left(\widetilde{H}_r^C\right)' \\ W & 0 & -\left(\widetilde{H}_{\widetilde{x}}^C\right)' \\ \widetilde{H}_{\widetilde{x}}^C & \widetilde{H}_r^C & 0 \end{bmatrix}$$

$$H_y^B = \begin{bmatrix} 0 & 0 & -\beta^{-1}\left(\widetilde{H}_r^F\right)' \\ 0 & 0 & -\beta^{-1}\left(\widetilde{H}_{\widetilde{x}}^F\right)' \\ \widetilde{H}_{\widetilde{x}}^B & 0 & 0 \end{bmatrix}$$

$$\Psi_y = \begin{bmatrix} 0 \\ 0 \\ \tilde{\Psi}_{\tilde{z}} \end{bmatrix}$$

$$\Psi_\mu = \begin{bmatrix} S' \\ 0 \\ 0 \end{bmatrix}$$

Since the structural form in (13) is isomorphic to (1), the rational expectations solution to the augmented model is:

$$y_t = B_y y_{t-1} + \sum_{i=0}^{\infty} F_y^i \Phi_y \mathbb{E}_t \widetilde{z}_{t+i} + \sum_{i=0}^{\infty} F_y^i \Phi_\mu \mathbb{E}_t \mu_{t+i} \tag{14}$$

where $F_y$ can be computed using a version of (3) and $\Phi_y$ and $\Phi_\mu$ are computed using (4).

To compute the results of a simulation or projection of the model that incorporates the effects of the instrument constraints, the multipliers $\mu_t$ $(t = 1, \dots)$ are used to impose the constraints, if necessary. When the instrument constraints do not bind in period $t$, the relevant elements of $\mu_t$ are zero. When the instrument constraints do bind, $\mu_t$ is chosen so that the instruments satisfy the constraints. The method solves for the values of $\mu_t, (t = 1, \dots)$ that impose the occasionally binding constraints.

The starting point for simulating the model with occasionally binding constraints is a 'baseline simulation' in which the constraints are ignored. A simulation over $H$ periods is produced by setting $\{\mu_t = 0\}_{t=1}^H$. From a given initial condition $x_0$ and a realization of the shocks $(\mathbb{E}_1\widetilde{z}_t, t = 1, \dots, H)$ the baseline simulation for $\{y_t\}_{t=1}^H$ is computed using (14) (with $\{\mu_t = 0\}_{t=1}^H$).

The baseline simulation can be checked to determine whether it violates the assumption that the constraints never bind. This amounts to checking whether $Sr_t > b$, $\forall t$. If any of these assumptions is violated in the baseline simulation, a quadratic programming procedure based on Holden and Paetz (2012) is used to enforce the occasionally binding constraints.

It is convenient to represent the occasionally binding constraints as inequality constraints on a set of 'target variables', $\tau$:

$$\tau_t = S_\tau y_t = Sr_t \tag{15}$$

The method to impose occasionally binding constraints is based on the insight that the effect of the $\mu$ shocks can be simply added to the baseline simulation, given the linearity of the solution. Inspection of (14) reveals that the effect of the fundamental $(\widetilde{z})$ and $\mu$ shocks enter linearly. To find the $\mu$ shocks that ensure that the target variables satisfy the OBC, requires finding a sequence of shocks that, when added to the baseline simulation, will achieve this. This requires knowledge of the impact of $\mu$ shocks in all periods $t = 1, \dots, H$ on the target variables in all periods $t = 1, \dots, H$.

Consider, then, the effects of the $\mu$ shocks $\{\mu_t\}_{t=1}^{H}$ on the endogenous variables in period 1 of the simulation. From (14), this is given by:

$$\widehat{y}_1 = \sum_{i=0}^{H-1} F^i \Phi_\mu \mu_{1+i}, \tag{16}$$

which captures the fact that in period 1 all of the shocks occur in (present and) future periods. The effects on the target variables are given by $\widehat{\tau}_1 = S_\tau \widehat{y}_1$.

For period 2, the rational expectations solution can be used to show that the effects on endogenous variables are:

$$\widehat{y}_2 = B_y \widehat{y}_1 + \sum_{i=0}^{H-2} F^i \Phi_\mu \mu_{2+i}, \tag{17}$$

and from the expression for $\widehat{y}_1$, this implies:

$$\widehat{y}_2 = B_y \sum_{i=0}^{H-1} F^i \Phi_\mu \mu_{1+i} + \sum_{i=0}^{H-2} F^i \Phi_\mu \mu_{2+i}. \tag{18}$$

This step provides a recursive scheme for building a matrix that maps the effects $\mu_t$ in periods $t = 1, \ldots, H$ to the target variables in each period. The first (block) row of this matrix can be found by expanding (16):

$$\widehat{\tau}_1 = \begin{bmatrix} S_\tau \Phi_\mu & \ldots & S_\tau F^{k-1} \Phi_\mu & \ldots & S_\tau F^{H-1} \Phi_\mu \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \\ \vdots \\ \mu_H \end{bmatrix} \tag{19}$$

The second row is built by using equation (18) to multiply the coefficients in the first row by $B_y$ and then adding the coefficients on shocks that arrive from period 2 onwards:

$$\widehat{\tau}_2 = \begin{bmatrix} S_\tau B_y \Phi_\mu & \ldots & S_\tau B_y F^{k-1} \Phi_\mu + S_\tau F^{k-2} \Phi_\mu & \ldots & S_\tau B_y F^{H-1} \Phi_\mu + S_\tau F^{H-2} \Phi_\mu \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \\ \vdots \\ \mu_H \end{bmatrix},$$

This can be applied for each row in turn and provides a method to write the mapping from the $\mu$ shocks to the target variables as:

$$\mathcal{T} = \mathcal{M} \mathcal{D} \tag{20}$$

where

$$\mathcal{T} = \begin{bmatrix} \widehat{\tau}_1 \\ \vdots \\ \widehat{\tau}_k \\ \vdots \\ \widehat{\tau}_H \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_k \\ \vdots \\ \mu_H \end{bmatrix}$$

where the rows of $\mathcal{M}$ are built using the recursive scheme described above.[11] The mapping (20) records the effects of all $\mu$ shocks on all target variables. This reflects the fact that there may be interactions

[11] If the number of OBCs is $n_\mu$, then $\tau$ and $\mu$ are $n_\mu \times 1$ vectors so that $\mathcal{T}$ and $\mathcal{D}$ are $(n_\mu H) \times 1$.

between the different constraints on individual policy instruments.

To incorporate the bounds on the OBCs, it is convenient to compute the vector $\widehat{\mathcal{T}}$ as the deviation of the target variables from their constraint values. This is done by recording the relevant rows of the baseline simulation $\{y_t\}_{t=1}^{H}$ and subtracting the value of the constraints. This normalization implies that if the baseline simulation was such that $\widehat{\mathcal{T}} > 0$, then that solution (which assumes that the constraints never bind) would be valid.

The optimal policy problem studied here is isomorphic to the one studied by Holden and Paetz (2012). So their insight that a quadratic programming problem can be use to solve for $\mathcal{D}$ applies. The quadratic programming problem is:

$$\min \quad \frac{1}{2}\mathcal{D}'\left(\mathcal{M}+\mathcal{M}'\right)\mathcal{D}+\widehat{\mathcal{T}}'\mathcal{D} \tag{21}$$

$$\text{subject to: } \widehat{\mathcal{T}}+\mathcal{M}\mathcal{D} \geq 0 \tag{22}$$

$$\mathcal{D} \geq 0 \tag{23}$$

The problem in equations (21)–(23) can be understood as follows. The constraint (22) ensures that the OBCs are respected. $\widehat{\mathcal{T}}$ is the baseline simulation for the target variables, measured relative to the constraint values. $\mathcal{M}\mathcal{D}=\mathcal{T}$ is the marginal effect of the $\mu$ shocks $\mathcal{D}$ on the target variables. So $\widehat{\mathcal{T}}+\mathcal{M}\mathcal{D}$ is the path of the target variables measured relative to their constraints after the $\mu$ shocks have been applied: requiring this to be non-negative implies that the constraints are respected.

The constraint (23) requires that the $\mu$ shock values used to impose the constraints are positive. This requirement ensures that the OBCs are truly binding.

Finally, note that the minimand (21) can be expanded as follows:

$$\begin{aligned}\frac{1}{2}\mathcal{D}'\left(\mathcal{M}+\mathcal{M}'\right)\mathcal{D}+\widehat{\mathcal{T}}'\mathcal{D} &= \frac{1}{2}\mathcal{D}'\mathcal{M}\mathcal{D}+\frac{1}{2}\mathcal{D}'\mathcal{M}'\mathcal{D}+\frac{1}{2}\widehat{\mathcal{T}}'\mathcal{D}+\frac{1}{2}\mathcal{D}'\widehat{\mathcal{T}}\\ &= \frac{1}{2}\mathcal{D}'\left(\mathcal{M}\mathcal{D}+\widehat{\mathcal{T}}\right)+\frac{1}{2}\left(\mathcal{M}\mathcal{D}+\widehat{\mathcal{T}}\right)'\mathcal{D},\end{aligned}$$

where the first line exploits the fact that $\widehat{\mathcal{T}}'\mathcal{D}$ is a scalar and the second line collects terms. The minimand is therefore analogous to a contemporary slackness condition: it achieves a minimum of zero when $\mathcal{D}=0$ or $\widehat{\mathcal{T}}+\mathcal{M}\mathcal{D}=0$.

The solution to the quadratic programming problem is a vector, $\mathcal{D}^*$. This vector represents the values of the anticipated $\mu$ values required for the OBCs on the policy instruments to be respected. The vectors of shocks $\mu$ can be extracted from $\mathcal{D}^*$ and the effects of these anticipated $\mu$ values can be incorporated into the simulation using (14).[12]

### 3.1.2 Fully optimal commitment

Under a fully optimal period-1 plan, the policymaker is unhindered by any past commitments. This means that $\lambda_0 = 0$ so that the first order conditions in period $t = 1$ satisfy:

$$0 = Qr_1 - \left(\widetilde{H}_r^C\right)'\lambda_1 - S'\mu_1 \tag{24}$$

$$0 = W\widetilde{x}_1 - \left(\widetilde{H}_{\widetilde{x}}^C\right)'\lambda_1 - \beta\left(\widetilde{H}_{\widetilde{x}}^B\right)'\mathbb{E}_1\lambda_2 \tag{25}$$

$$0 = \widetilde{H}_{\widetilde{x}}^F\mathbb{E}_1\widetilde{x}_2 + \widetilde{H}_{\widetilde{x}}^C\widetilde{x}_1 + \widetilde{H}_{\widetilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^F\mathbb{E}_t r_2 + \widetilde{H}_r^C r_1 - \widetilde{\Psi}_{\widetilde{z}}\widetilde{z}_1 \tag{26}$$

$$0 = \mu_1'\left(Sr_1 - b\right) \tag{27}$$

[12] The first $H$ elements of $\mathcal{D}^*$ correspond to the values of the first element of $\mu$ for periods $t = 1, \ldots, H$ and so on.

which can be written as:

$$H_y^F \mathbb{E}_1 y_2 + H_y^C y_1 + H_{y,1}^B y_0 = \Psi_y \widehat{z}_1$$

where

$$H_{y,1}^B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \widetilde{H}_{\widetilde{x}}^B & 0 & 0 \end{bmatrix}$$

The solution for $y_2$ satisfies the rational expectations solution, (14), since $\lambda_1$ does constrain period 2 policy. This means that:

$$\begin{aligned} y_1 &= -\left(H_y^C + H_y^F B_y\right)^{-1} H_{y,1}^B y_0 + \left(H_y^C + H_y^F B_y\right)^{-1} \Psi_y \widehat{z}_1 - \left(H_y^C + H_y^F B_y\right)^{-1} H_y^F \sum_{i=0}^{\infty} F_y^i \Phi_y \mathbb{E}_1 \widehat{z}_{2+i} \\ &= -\left(H_y^C + H_y^F B_y\right)^{-1} H_{y,1}^B y_0 + \Phi_y \widehat{z}_1 + F_y \sum_{i=0}^{\infty} F_y^i \Phi_y \mathbb{E}_1 \widehat{z}_{2+i} \end{aligned}$$

Employing the same partitioning of the $\widehat{z}$ vector used above gives:

$$y_1 = B_{y,1} y_0 + \sum_{i=0}^{\infty} F_y^i \Phi_{\widetilde{z}} \mathbb{E}_1 \widetilde{z}_{1+i} + \sum_{i=0}^{\infty} F_y^i \Phi_{\mu} \mathbb{E}_1 \mu_{1+i} \tag{28}$$

where

$$B_{y,1} \equiv -\left(H_y^C + H_y^F B_y\right)^{-1} H_{y,1}^B$$

Equation (28) shows that the solution in period $t = 1$ has the same generic form as (14) with $B_{y,1}$ in place of $B_y$. This demonstrates that the effects of anticipated disturbances on period 1 outcomes are identical to those used to compute the $\mathcal{M}$ matrix described in Section 3.1.1. Since the solution for outcomes in periods $t \geq 2$ is identical to the timeless perspective case, the only difference in constructing a simulation for an optimal period-1 plan is to use (28) in place of (14) to compute the outcomes in the first period of the simulation.[13]

## 3.2 Discussion

The building blocks of the method presented above are the optimal commitment approach presented by Dennis (2007), the application of the Anderson and Moore (1985) algorithm to derive a representation including anticipated disturbances and the quadratic programming method developed by Holden and Paetz (2012) to use anticipated disturbances to impose inequality constraints.

As argued by Holden and Paetz (2012), an advantage of the quadratic programming approach is that it easily handles multiple constraints. In the context of the present paper, the method is therefore readily applicable to cases in which there are multiple policy instruments. Such cases are of substantial interest given the increased use of unconventional monetary policy tools (in part prompted by prolonged periods in which the short-term policy rate has been constrained by a lower bound) and a growing body of research exploring coordination of macroeconomic policies (including monetary, macro-prudential and fiscal).

Holden and Paetz (2012) also provide a general discussion of uniqueness and existence of an equilibrium that satisfies the occasionally binding constraints.[14] Holden (2019) provides a more extensive analysis, with a particular focus on New Keynesian models with a lower bound constraint on the mon-

[13]Indeed, if $y_0 = 0$, the timeless perspective solution coincides with the optimal period-1 plan.

[14]For example, a sufficient condition for the quadratic programming problem to have a unique solution is if the matrix $(\mathcal{M} + \mathcal{M}')$ is positive semi-definite.

etary policy rate. Since the method presented in this section is designed for optimal policy analysis using general (potentially large-scale) models, few of the specific results considered in previous papers will apply.[15]

The method presented in this section has many similarities with the 'OccBin' approach developed by Guerrieri and Iacoviello (2015). The equilibrium concept studied by Guerrieri and Iacoviello (2015) is identical and so the results from applying that approach will coincide with those presented here.

In practice, the approach presented here has three potential advantages over the OccBin toolkit. First, OccBin is not designed to incorporate optimal policy behavior, which means that the model must be rendered in an appropriate form before using the OccBin toolkit. While this is clearly feasible (see, for example, Canzoneri et al., 2020), doing so for a large-scale model represents a substantial undertaking.

Second, as noted above, the method presented here scales readily to the imposition of bounds on multiple policy instruments. Incorporating $N$ occasionally binding constraints using OccBin requires specifying $2^{n_\mu}$ alternative sets of model equations, whereas the Holden and Paetz (2012) approach requires $n_\mu$ additional 'shocks' (i.e., the multipliers, $\mu$). This consideration is particularly important given the reliance of the OccBin approach on a 'guess and verify' method for finding the equilibrium. If the model may be in any of $2^{n_\mu}$ possible 'states' *in each period* $t$, the set of possible equilibria to be checked becomes very large.[16]

Finally, the algorithm is designed to incorporate non-zero 'anticipated disturbances' ($\{z_{t+i}\}_{i=1}^{H}$) and can therefore be used for optimal policy analysis around a non-model-based forecast or scenario.

### 3.3 Example

The example is adapted from Harrison (2012) and considers optimal commitment policy in a model with quantitative easing (QE). The model incorporates simple portfolio frictions such that the relative yields on short-term and long-term government debt depend on the relative quantities of these assets held by households. Asset purchases by the central bank (i.e., QE) can affect the relative bond holdings of household and hence long-term bond rates. As well as incorporating the zero lower bound on the nominal interest rate, a non-negativity constraint and an upper bound on the quantitative easing instrument are also imposed.

The model equations are:

$$\widehat{x}_t - \eta\widehat{x}_{t-1} = \mathbb{E}_t\left(\widehat{x}_{t+1} - \eta\widehat{x}_t)\right) - \sigma\left[\frac{1}{1+\delta}\widehat{R}_t + \frac{\delta}{1+\delta}\widehat{R}^e_{L,t} - \mathbb{E}_t\widehat{\pi}_{t+1} - \widehat{r}^*_t\right] \tag{29}$$

$$\widehat{R}^e_{L,t} = \widehat{R}_t - \nu q_t \tag{30}$$

$$\widehat{\mathcal{R}}_t = \chi\beta\mathbb{E}_t\widehat{\mathcal{R}}_{t+1} + (1-\chi\beta)\,\widehat{R}^e_{L,t} \tag{31}$$

$$\widehat{\pi}_t = \beta\mathbb{E}_t\widehat{\pi}_{t+1} + \kappa\widehat{x}_t - \frac{\kappa\eta}{1+\psi\sigma}\widehat{x}_{t-1} \tag{32}$$

$$\widehat{r}^*_t = \rho\widehat{r}^*_{t-1} + \varepsilon_t \tag{33}$$

where the '$\widehat{\ }$' notation denotes log-deviations from steady state, scaled by 100.[17] The output gap is expressed as a percentage deviation of output ($y$) from potential output ($y^*$), $\widehat{x}_t = 100\ln\left(y_t/y^*_t\right)$. The quantitative easing variable, $q$, represents the fraction of long-term government debt held by the central bank and is unscaled.

[15] Extensive experiments have revealed few indications of equilibrium non-existence or multiplicity. An intriguing possibility is that monetary policy under commitment in New Keynesian models tends to generate policy behavior with a strong 'price level targeting' element. Holden (2019) argues that price level targeting rules are less likely to exhibit multiple equilibria in New Keynesian models with a lower bound on the policy rate.

[16] Of course, the speed and efficiency of the quadratic programming approach will also depend on the precise algorithm used to compute the solution.

[17] For a variable $z_t$, $\widehat{z}_t \equiv 100\ln\left(z_t/z_{ss}\right)$ where $z_{ss}$ is the steady state value of the variable.

Equation (29) is an aggregate demand relationship derived from the Euler equation of optimizing households. The output gap $\widehat{x}$ evolves according to a standard Euler equation, with inertial terms appearing due to external habit formation (when $\eta > 0$). The portfolio frictions imply that the interest rate that drives spending decisions is a weighted average of the short-term policy rate $\widehat{R}$ and the one-period return on the long-term bond, $R^e_{L,t}$.

Equation (30) is a no-arbitrage relationship between the one-period return on the long-term bond and the short-term interest rate. Returns are equalized up to an expression in $q$. QE affects relative bond yields because deviations from households' preferred portfolio mix generate utility losses (Harrison, 2012) or portfolio adjustment costs (Harrison, 2017). Given the stylized treatment of fiscal policy introduced by Harrison (2017), the central bank can control households' portfolio mix directly through its QE actions.

Equation (31) is a pricing equation for the yield to maturity on the long-term bond, which is an infinitely-lived consol with a geometrically declining coupon.

Equation (32) is a Phillips curve relating the output gap to inflation and can be derived from the assumption of Calvo (1983) price stickiness. Finally, equation (33) describes the evolution of the natural real interest rate, $r^*$, as a simple AR(1) process.

The loss function is given by:

$$\mathcal{L}_t = \sum_{i=0}^{\infty} \beta^i \left[ \widehat{\pi}^2_{t+i} + \lambda_x \widehat{x}^2_{t+i} + \lambda_{\Delta R} \left( \widehat{R}_{t+i} - \widehat{R}_{t+i-1} \right)^2 + \lambda_q q^2_{t+i} + \lambda_{\Delta q} \left( q_{t+i} - q_{t+i-1} \right)^2 \right] \tag{34}$$

which says that the policymaker attempts to stabilize inflation, the output gap and movements in its policy instruments. The loss function penalizes changes in the short-term nominal interest rate and the QE instrument to capture a preference for gradualism in policymaking.[18] The loss function also includes a penalty for the use of the quantitative easing instrument (with weight $\lambda_q$) since models with explicit portfolio frictions (for example Harrison, 2012, 2017) predict that these frictions generate welfare costs.[19]

Minimization of the loss function is subject to bounds on the policy instruments:

$$\widehat{R}_t \geq 100 \ln \beta \tag{35}$$

$$q_t \geq 0 \tag{36}$$

$$q_t \leq \bar{q} \leq 1 \tag{37}$$

Constraint (35) is the familiar zero lower bound constraint on the policy rate, expressed in terms of the log deviation of the policy rate from steady-state, $\widehat{R}$. Constraint (36) states that the central bank holds non-negative quantities of long-term government debt (or equivalently that the central bank may not issue long-term debt that is perfectly substitutable for long-term government bonds). Constraint (37) says that the central bank is limited in the fraction of long-term government debt that it may hold. There is a maximal upper bound of 1, since the central bank may not purchase more than the entire stock of outstanding long-term bonds. However, the upper bound $\bar{q}$ may be less than 1, reflecting, for example, concerns about interest rate risk exposure associated with a large central bank balance sheet (Harrison, 2017).

The parameter values are shown in Table 1. These values are taken from Harrison (2012) and Harrison (2017) where possible. The weights on the output gap, inflation and the policy rate capture a 'balanced' loss function sometimes considered by monetary policymakers (for example, Yellen, 2012;

[18] Including terms that penalize the variability of policy instruments is widely adopted in the literature studying optimal policy and comparing alternative policy rules in estimated models. See, for example: Rudebusch and Svensson (1999); Levin and Williams (2003); Givens (2012). One rationale for including a term penalizing changes in the policy rate is to avoid destabilizing effects in financial markets (Lowe et al., 1997).

[19] The weights in the loss function are set with reference to applications by policymakers (for example, Yellen, 2012; Carney, 2017), rather than microfoundations.

Carney, 2017). The weights on QE and the change in QE are set so that the initial response of QE to the shock considered below mimics the scale of asset purchases undertaken in the United Kingdom in 2009. Similarly, the parameter governing the strength of QE, $\nu$, is set such that the response of the long-term bond rate in the simulation below is similar to that estimated effect of QE in the United Kingdom. These results are discussed further below.

| Parameter | Description | Value |
|---|---|---|
| $\beta$ | Household discount factor | 0.9925 |
| $\kappa$ | Slope of Phillips curve | 0.024 |
| $\psi$ | Inverse Frisch elasticity | 0.11 |
| $\sigma$ | Elasticity of intertemporal substitution | 1 |
| $\delta$ | Share of long-term to short-term debt | 0.3 |
| $\chi$ | Long bond coupon decay rate | 0.98 |
| $\rho$ | Autocorrelation of natural real interest rate | 0.85 |
| $\nu$ | Elasticity of long-term bond rate to QE | 1.25 |
| $\eta$ | Habit formation parameter | 0.8 |
| $\bar{q}$ | Upper bound on QE | 0.5 |
| $\lambda_x$ | Loss function weight on output gap | 0.25 |
| $\lambda_{\Delta R}$ | Loss function weight on policy rate smoothing | 1 |
| $\lambda_q$ | Loss function weight on QE | 0.05 |
| $\lambda_{\Delta q}$ | Loss function weight on change in QE | 5 |

**Table 1:** Parameter values for QE model

The experiment considers a large reduction in $r^*$ that causes the short-term policy rate to be constrained by the zero bound. The shock is calibrated so that the natural rate falls from its steady state level of 2% per year to -8%. Policy is set under commitment with a 'timeless perspective' solution.

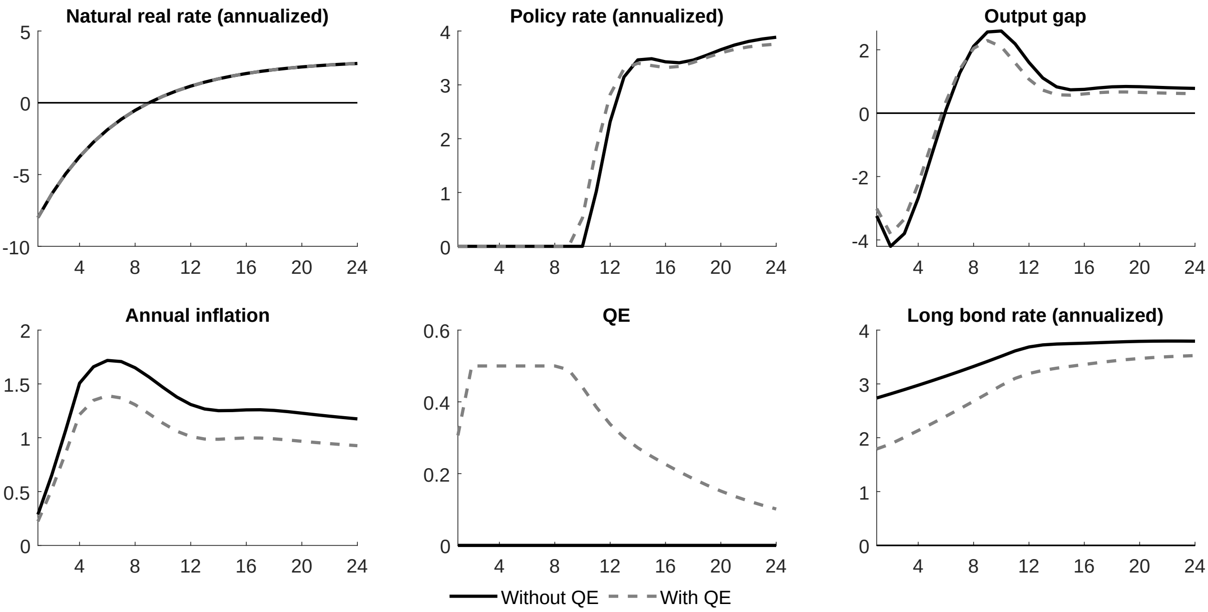


**Figure 1:** Recessionary shock under optimal commitment with instrument bounds

Figure 1 shows the results of this simulation. Each panel shows the results from two versions of the simulation. The solid lines show the case in which the QE instrument is unavailable.[20] The dashed lines

[20]This case can be interpreted as the case in which QE is infinitely costly $\lambda_q \to \infty$. In practice, it is implemented

show the case in which QE is used, with the loss function specification described above.

Focusing first on the case in which QE is unavailable (solid lines), the fall in the natural rate of interest, $r^*$ is sufficient to push the policy rate to the zero bound immediately – a cut of three percentage points from its steady-state level of 3%. The policy rate stays at the zero bound for 10 quarters, slightly longer than the spell for which the natural real interest rate is negative. While this 'lower for longer' behavior is insufficient to prevent a near term recession, it creates the conditions for a prolonged future boom.

The persistent boom generated by the optimal policy response is sufficient to generate an immediate *rise* in inflation that persists for many years. The long-lived increase in inflation reduces expected real interest rates, cushioning the effects of the shock on output. The stimulus of the promise requires the policy rate to rise rapidly after liftoff from the ZLB and quickly rise above the steady-state level of 3%. This behavior implies that the long-term bond rate declines only marginally below the steady-state level of 3% before rising persistently above.

When QE is used (dashed lines), the policymaker immediately purchases around 30% of the long-term government debt stock, with QE rising to its upper bound of 50% within a few quarters. QE remains at its upper bound for several quarters, before gradually unwinding. The QE response lowers the long-term bond rate by around 100 basis points on impact. The scale of the initial QE response is broadly in line with QE1 in the United Kingdom (Daines et al., 2012) and the decline in the long-term bond rate is similar to the estimated effect of QE1 in Joyce et al. (2011).

The reduction in long-term bond rates induced by QE stimulates spending, thus reducing the scale of the initial recession relative to the case in which QE is unavailable (solid lines). Since QE has an immediate effect on spending, there is less need to reduce real interest rates via an increase in inflation expectations. The subsequent boom in output and inflation overshoot are both smaller than the case in which QE is unavailable. The stimulus from QE permits a slightly earlier liftoff from the ZLB.

# 4 Optimal discretion with 'anticipated disturbances'

The contribution of this section is to characterize the solution of the model under optimal time-consistent policy when agents in period $t$ may anticipate non-zero future disturbances: $\widetilde{z}_{t+s} \neq 0$ , $s > 0$. These shocks are labeled as 'anticipated disturbances', though as discussed in Section 2 they are sometimes called 'news shocks' in the literature. As noted in the introduction, including anticipated disturbances is a key requirement for methods that can be applied to forecasts or scenarios that are not constructed using the model alone.

The analysis in this section assumes that the policy instruments are not subject to bounds and therefore represents a stepping stone to the cases in which policy instruments are constrained, analyzed in Sections 5 and 6.2.

## 4.1 Method

The policymaker minimizes the loss function (6), repeated here for convenience:

$$\begin{aligned} \min_{x_t, r_t} \mathcal{L}_t &= \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ (\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + (r_{t+i})' Q (r_{t+i}) \right\} \\ &= (\widetilde{x}_t)' W (\widetilde{x}_t) + (r_t)' Q (r_t) + \beta \mathcal{L}_{t+1} \end{aligned} \quad (38)$$

where the recursive representation of the loss function (38) is convenient for the subsequent analysis.

by including a 'rule' for QE given by $q_t = 0$ and assuming that the only instrument available to the policymaker is the short-term interest rate.

Minimization is subject to the constraint imposed by the partitioned model equations (5):

$$\widetilde{H}^F_{\tilde{x}}\mathbb{E}_t\widetilde{x}_{t+1} + \widetilde{H}^C_{\tilde{x}}\widetilde{x}_t + \widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} + \widetilde{H}^F_r\mathbb{E}_t r_{t+1} + \widetilde{H}^C_r r_t = \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t$$

The solution to be found will be of the following form:

$$\widetilde{x}_t = B_{\tilde{x}\tilde{x}}\widetilde{x}_{t-1} + \sum_{s=0}^{H}\left(F_{s,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{s,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{t+s} \tag{39}$$

$$r_t = B_{r\tilde{x}}\widetilde{x}_{t-1} + \sum_{s=0}^{H}\left(F_{s,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{s,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{t+s} \tag{40}$$

where: $B_{\tilde{x}\tilde{x}}$ is a matrix of loadings on lagged endogenous variables in the law of motion for the endogenous variables; $B_{r\tilde{x}}$ the loadings on lagged endogenous variables in the law of motion for the instruments; $F_{s,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}$ the loadings on s-period ahead shocks in the law of motion for the endogenous variables arising via their effect on the endogenous variables; $F_{s,\tilde{x}r}\Phi_{r\tilde{z}}$ the loadings on s-period ahead shocks in the law of motion for the endogenous variables arising via their effect on the instruments; $F_{s,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}$ the loadings on s-period ahead shocks in the law of motion for the instruments arising via their effect on the endogenous variables; $F_{s,rr}\Phi_{r\tilde{z}}$ the loadings on s-period ahead shocks in the law of motion for the instruments arising via their effect on the instruments; and where $F_{0,\tilde{x}\tilde{x}} = \mathbb{I}$, $F_{0,\tilde{x}r} = 0$, $F_{0,r\tilde{x}} = 0$ and $F_{0,rr} = \mathbb{I}$.

The derivation of the solution will characterize the forward loading coefficients $\{F_{s,\tilde{x}\tilde{x}}\}_{s=1}^{H}$, $\{F_{s,\tilde{x}r}\}_{s=1}^{H}$, $\{F_{s,r\tilde{x}}\}_{s=1}^{H}$ and $\{F_{s,rr}\}_{s=1}^{H}$. In doing so, it is demonstrated that $B_{\tilde{x}\tilde{x}}$, $B_{r\tilde{x}}$, $\Phi_{\tilde{x}\tilde{z}}$ and $\Phi_{r\tilde{z}}$ are unaffected by the presence of anticipated disturbances. Recursive formula that allow the forward loading coefficients to be computed up to an arbitrary $H$ are also derived.[21]

The algorithm works by backward induction, leveraging the fact that the solution in period $t + H$ is identical to that derived in the absence of anticipated disturbances. For completeness, a notationally-consistent version of the Dennis (2007) algorithm (without anticipated disturbances) is presented in Appendix A. The environment is perfect foresight, so the expectations operator is omitted throughout.

Appendix B presents the backward induction steps that derive the solutions for periods $H$, $H-1$, $H-2$ and $H-3$. As noted above, that derivation demonstrates that the loadings on $\widetilde{x}_{t-1}$ (that is $B_{\tilde{x}\tilde{x}}$ and $B_{r\tilde{x}}$) are unaffected by the presence of anticipated disturbances.

The backward induction process shows that the law of motion for the equilibrium under optimal discretion with anticipated disturbances up to a horizon of $H$ is:

$$\begin{aligned} x_t &= Bx_{t-1} + \sum_{s=0}^{H} F_s\Phi_{\tilde{z}}\widetilde{z}_{t+s} \\ \begin{bmatrix}\widetilde{x}_t \\ r_t\end{bmatrix} &= \begin{bmatrix}B_{\tilde{x}\tilde{x}} & 0 \\ B_{r\tilde{x}} & 0\end{bmatrix}\begin{bmatrix}\widetilde{x}_{t-1} \\ r_{t-1}\end{bmatrix} + \sum_{s=0}^{H}\begin{bmatrix}F_{s,\tilde{x}\tilde{x}} & F_{s,\tilde{x}r} \\ F_{s,r\tilde{x}} & F_{s,rr}\end{bmatrix}\begin{bmatrix}\Phi_{\tilde{x}\tilde{z}} \\ \Phi_{r\tilde{z}}\end{bmatrix}\widetilde{z}_{t+s}\end{aligned} \tag{41}$$

The $B$ and $\Phi$ matrices are given by:

$$\begin{aligned} B_{\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r B_{r\tilde{x}}\right) \\ \Phi_{\tilde{x}\tilde{z}} &= \Theta^{-1}\left(\Psi_{\tilde{z}} - \widetilde{H}^C_r\Phi_{r\tilde{z}}\right) \\ B_{r\tilde{x}} &= \Delta_r^{-1}\Delta_{\tilde{x}} \\ \Phi_{r\tilde{z}} &= \Delta_r^{-1}\Delta_{\tilde{z}}\end{aligned}$$

[21] A convenient implication of these two results is that the forward loadings required for a particular projection can be 'post-computed' without other parts of the optimal discretion solution needing to be recomputed.

where:

$$\begin{aligned}\Theta &= \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}} B_{\tilde{x}\tilde{x}} + \widetilde{H}^F_r B_{r\tilde{x}} \\ \Delta_r &= Q + \zeta \widetilde{H}^C_r \\ \Delta_{\tilde{x}} &= -\zeta \widetilde{H}^B_{\tilde{x}} \\ \Delta_{\tilde{z}} &= \zeta \widetilde{\Psi}_{\tilde{z}}\end{aligned}$$

and:

$$\zeta = \left(\Theta^{-1}\widetilde{H}^C_r\right)' (W + \beta V_{\tilde{x}\tilde{x}})\, \Theta^{-1}$$

with:

$$V_{\tilde{x}\tilde{x}} = (B_{\tilde{x}\tilde{x}})' W B_{\tilde{x}\tilde{x}} + (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta\, (B_{\tilde{x}\tilde{x}})' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}}$$

which captures the marginal effect of the endogenous variables ($\widetilde{x}$) on future losses.

Appendix B shows that the forward loadings in (41), $F_s \equiv \begin{bmatrix} F_{s,\tilde{x}\tilde{x}} & F_{s,\tilde{x}r} \\ F_{s,r\tilde{x}} & F_{s,rr} \end{bmatrix}$, can be constructed recursively as follows:

$$F_s = F^{ps}_s + F^{pol}_s \tag{42}$$

where $F^{ps}_s$ and $F^{pol}_s$ measure the impact of $s$-period ahead anticipated disturbances via their effect on private sector behaviour and policy optimisation, defined as:

$$\begin{aligned}F^{ps}_s &= \mathcal{F}^{ps} F_{s-1} \\ F^{pol}_s &= \mathcal{F}^{pol} \Sigma_s\end{aligned}$$

where $F_0 = \mathbb{I}$.

The matrix $\mathcal{F}^{ps}$ measures the impact of anticipated disturbances via one-period ahead expectations on private sector behavior (taking into account the action of policy) and can be decomposed as follows:

$$\mathcal{F}^{ps} = \begin{bmatrix} \mathcal{F}^{ps}_{\tilde{x}\tilde{x}} & \mathcal{F}^{ps}_{\tilde{x}r} \\ \mathcal{F}^{ps}_{r\tilde{x}} & \mathcal{F}^{ps}_{rr} \end{bmatrix}$$

where:

$$\begin{aligned}\mathcal{F}^{ps}_{\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}^F_{\tilde{x}} + \widetilde{H}^C_r \Delta_r^{-1} \Delta_{F^{ps}_{r\tilde{x}}}\right) \\ \mathcal{F}^{ps}_{\tilde{x}r} &= -\Theta^{-1}\left(\widetilde{H}^F_r + \widetilde{H}^C_r \Delta_r^{-1} \Delta_{F^{ps}_{rr}}\right) \\ \mathcal{F}^{ps}_{r\tilde{x}} &= \Delta_r^{-1} \Delta_{F^{ps}_{r\tilde{x}}} \\ \mathcal{F}^{ps}_{rr} &= \Delta_r^{-1} \Delta_{F^{ps}_{rr}}\end{aligned}$$

and:

$$\begin{aligned}\Delta_{F^{ps}_{r\tilde{x}}} &= -\zeta H^F_{\tilde{x}} \\ \Delta_{F^{ps}_{rr}} &= -\zeta \widetilde{H}^F_r\end{aligned}$$

The matrix $\mathcal{F}^{pol}$ measures the impact of anticipated disturbances via one-period ahead losses on policy optimization (taking into account private sector behavior) and has the following structure:

$$\mathcal{F}^{pol} = \begin{bmatrix} \mathcal{F}^{pol}_{\tilde{x}\tilde{x}} & \mathcal{F}^{pol}_{\tilde{x}r} \\ \mathcal{F}^{pol}_{r\tilde{x}} & \mathcal{F}^{pol}_{rr} \end{bmatrix}$$

where:

$$
\begin{aligned}
\mathcal{F}^{pol}_{\tilde{x}\tilde{x}} &= -\Theta^{-1}\widetilde{H}^{C}_{r}\Delta^{-1}_{r}\Delta_{F^{pol}_{r\tilde{x}}} \\
\mathcal{F}^{pol}_{\tilde{x}r} &= -\Theta^{-1}\widetilde{H}^{C}_{r}\Delta^{-1}_{r}\Delta_{F^{pol}_{rr}} \\
\mathcal{F}^{pol}_{r\tilde{x}} &= \Delta^{-1}_{r}\Delta_{F^{pol}_{r\tilde{x}}} \\
\mathcal{F}^{pol}_{rr} &= \Delta^{-1}_{r}\Delta_{F^{pol}_{rr}}
\end{aligned}
$$

and:

$$
\begin{aligned}
\Delta_{F^{pol}_{r\tilde{x}}} &= \left(\Theta^{-1}\widetilde{H}^{C}_{r}\right)' \beta \left(B_{\tilde{x}\tilde{x}}\right)' \\
\Delta_{F^{pol}_{rr}} &= \left(\Theta^{-1}\widetilde{H}^{C}_{r}\right)' \beta \left(B_{r\tilde{x}}\right)'
\end{aligned}
$$

The matrix $\Sigma_s$ describes the cumulated impact of anticipated disturbances $s$ periods ahead on one-period ahead losses and can be computed recursively as:

$$
\Sigma_s = \beta B' \Sigma_{s-1} + \Omega F_{s-1}
$$

where $\Sigma_0 = 0$ and:

$$
\Omega = \begin{bmatrix} W + \beta V_{\tilde{x}\tilde{x}} & 0 \\ 0 & Q \end{bmatrix}
$$

## 4.2 Discussion

As noted above, the matrices describing the autoregressive behavior of the endogenous variables and the contemporaneous effects of shocks (that is, $B_{\tilde{x}\tilde{x}}$, $B_{r\tilde{x}}$, $\Phi_{\tilde{x}\tilde{z}}$ and $\Phi_{r\tilde{z}}$) are identical to those derived by Dennis (2007) for the case where there are no anticipated disturbances. The key innovation therefore relates to the incorporation of the effects of anticipated disturbances on the decisions of the policymaker and private agents.

Equation (42) shows that the loading coefficients on anticipated disturbances arising via both private sector expectations and policy optimization depend on the coefficients on future disturbances arising from both sources. That is, the private sector correctly takes into account that (e.g.) two-period-ahead anticipated disturbances affect one-period-ahead policy optimization and the policymaker correctly takes into account that (e.g.) two-period-ahead anticipated disturbances affect one-period-ahead private sector behavior via both their own expectations and their rational understanding of how policy will respond.

These observations indicate that the Dennis (2007) solution derived under the assumption of no anticipated disturbances will remain valid in the presence of such disturbances if it gives rise to a static targeting criterion, whereby the first order condition to the optimal policy problem does not depend on endogenous state variables.

## 4.3 Example

The example uses the model developed by Ferrero et al. (2018) to analyze monetary and macro-prudential policies. The model features a household sector with distinct borrowers and savers and a standard New Keynesian sticky price specification.

Ferrero et al. (2018) derive a log-linear approximation around an efficient steady-state. The log-linearized model equations are:[22]

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} \tag{43}$$

$$x_t - \xi \tilde{c}_t = -\sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1}) + \mathbb{E}_t(x_{t+1} - \xi \tilde{c}_{t+1}) \tag{44}$$

$$d_t^b = \gamma_d \left(d_{t-1}^b - \pi_t\right) + (1-\gamma_d)\left(q_t + (1-\xi)\,\tilde{h}_t\right) \tag{45}$$

$$d_t^b = \frac{1}{\beta_s}(i_{t-1} + d_{t-1}^b - \pi_t) + \frac{1-\xi}{\Theta}(\tilde{h}_t - \tilde{h}_{t-1}) + \frac{1-\xi}{\eta}\tilde{c}_t \tag{46}$$

$$q_t = \frac{1+\tau^h - \beta_s}{1+\tau^h}\left(\sigma_h \xi \tilde{h}_t + u_t^h\right) + \sigma x_t - \sigma \xi \tilde{c}_t + \frac{\beta_s}{1+\tau^h}\mathbb{E}_t\left(\sigma \xi \tilde{c}_{t+1} - \sigma x_{t+1} + q_{t+1}\right) \tag{47}$$

$$\begin{aligned} q_t = {} & \frac{(1-\gamma_d)\,\tilde{\mu}\Theta}{1-(1-\gamma_d)\,\tilde{\mu}\Theta}\mu_t - \frac{1-(1-\gamma_d)\,\tilde{\mu}\Theta - \beta_b}{1-(1-\gamma_d)\,\tilde{\mu}\Theta}\left[\sigma_h (1-\xi)\,\tilde{h}_t - u_t^h\right] \\ & + \sigma(1-\xi)\,\tilde{c}_t + \sigma x_t + \frac{\beta_b}{1-(1-\gamma_d)\,\tilde{\mu}\Theta}\mathbb{E}_t\left(q_{t+1} - \sigma(1-\xi)\,\tilde{c}_{t+1} - \sigma x_{t+1}\right) \end{aligned} \tag{48}$$

$$\begin{aligned} x_t + (1-\xi)\,\tilde{c}_t = {} & \mathbb{E}_t\left(x_{t+1} + (1-\xi)\,\tilde{c}_{t+1}\right) + \sigma^{-1}\mathbb{E}_t \pi_{t+1} - \sigma^{-1}\frac{\beta_b}{\beta_s(1-\tilde{\mu})} i_t \\ & - \frac{\tilde{\mu}}{\sigma(1-\tilde{\mu})}\mu_t + \frac{\beta_b \gamma_d \tilde{\mu}}{\sigma(1-\tilde{\mu})}\mathbb{E}_t \mu_{t+1} \end{aligned} \tag{49}$$

$$u_t^h = \rho_h u_{t-1}^h + \varepsilon_t^h \tag{50}$$

Equation (43) is a standard New Keynesian Phillips curve, relating inflation, $\pi$ to the output gap $x$. Equation (44) is derived from the Euler equation of savers and represents the 'IS' curve of the model. As in a standard representative household model, the output gap depends on the expected output gap and the ex-ante real interest rate ($i_t - \mathbb{E}_t \pi_{t+1}$, where $i$ is the short-term nominal interest rate). However, in this model, the IS curve also depends on the difference between the consumption level of borrowers and savers: the 'consumption gap' $\tilde{c}_t \equiv c_t^b - c_t^s$ where $c_t^b$ and $c_t^s$ denote the log-deviations of borrower consumption and saver consumption from steady state.

Equation (45) is the borrowing constraint in the economy, which states that the debt of borrowers depends on debt in the previous period and a loan-to-value constraint that depends on the real house price, $q_t$ and the housing stock, which can be represented as a function of the 'housing gap' $\tilde{h}_t \equiv h_t^b - h_t^s$. Equation (46) is the borrower's budget constraint (evaluated in equilibrium) which provides a law of motion for debt.

Equations (47) and (48) are derived from the housing demand equations for savers and borrowers respectively. In each case, the equation has a familiar Euler equation form, depending on the housing gap, expected demand (output and consumption gaps) and expected house prices. The demand for housing for borrowers also depends on $\mu_t$, the Lagrange multiplier on the borrowing constraint. The multiplier on the borrowing constraint also appears in the Euler equation for borrower's consumption, (49).

Finally equation (50) describes the evolution of the exogenous housing demand disturbance used in the example. The process has a familiar autoregressive structure. It is usually assumed that the shock component $\varepsilon^h$ is an identically and independently distributed mean zero random variable, so that that $\mathbb{E}_t \varepsilon_{t+h} = 0$, $\forall h > 0$. However, the example will consider a fully anticipated movement in $\varepsilon_{t+h}$ for $h > 0$: a 'news shock'.

Each time period in the model is interpreted as a quarter (of a year). The deep parameter values,

[22] The model is simplified slightly by ignoring preference and cost-push shocks and ignoring transitory variations in the loan to value ratio and capital requirements (which are considered as macro-prudential policy instruments by Ferrero et al. (2018)).

following Ferrero et al. (2018), are shown in Table 2.[23]

| | Description | Value |
|---|---|---|
| $\beta_s$ | Saver discount factor | 0.9925 |
| $\sigma$ | Inverse elasticity of substitution (consumption) | 1 |
| $\varphi$ | Inverse Frisch elasticity | 1 |
| $\gamma_d$ | Debt limit inertia | 0.7 |
| $\Theta$ | Debt limit (fraction of house value) | 0.9 |
| $\gamma$ | Slope of Phillips curve | 0.024 |
| $\beta_b$ | Borrower discount factor | 0.99 |
| $\xi$ | Fraction of borrowers in economy | 0.57 |
| $\eta$ | Debt to (quarterly) GDP ratio | 1.8 |
| $\psi$ | Elasticity of funding cost to capital ratio | 0.05 |
| $\sigma_h$ | Inverse elasticity of substitution (housing) | 25 |
| $\varepsilon$ | Elasticity of substitution between final output varieties | 6 |
| $\rho_h$ | Housing demand shock persistence | 0.9 |

**Table 2:** Parameter Values

To illustrate the method, the simulation is repeated for two specifications of the loss function:

$$\mathcal{L}_0^{FIT} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_s^t \left( x_t^2 + \lambda_\pi \pi_t^2 \right) \tag{51}$$

$$\mathcal{L}_0^{LAW} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_s^t \left( x_t^2 + \lambda_\pi \pi_t^2 + \lambda_c \tilde{c}_t^2 + \lambda_h \tilde{h}_t^2 \right) \tag{52}$$

Following Ferrero et al. (2018), (51) is interpreted as a 'flexible inflation targeting' ('FIT') loss function, corresponding to the generic mandate that applies to many central banks. This requires the central bank to stabilize variations in inflation and economic activity (the output gap).

Again following Ferrero et al. (2018), equation (52) is interpreted as a case in which the monetary policy maker is 'leaning against the wind' (LAW). In this case the monetary policy instrument (the nominal interest rate) is used to stabilize fluctuations in the consumption and housing gaps, alongside inflation and the output gap. Since consumption and housing gaps arise from the financial frictions in the model, this specification is one in which the *monetary policy* mandate includes financial stability considerations.[24]

The results are compared to a 'naive' application of the Dennis (2007) algorithm that does not incorporate the effects of anticipated disturbances. Specifically, that approach consists of the following steps:

1. Solve the model using the Dennis (2007) algorithm (as in Appendix A). This gives a solution of the form $x_t = B_D x_{t-1} + \Phi_D z_t$.

2. Use the first order condition of the Dennis (2007) algorithm to substitute for the policy rules in

[23]The additional composite parameters in the log-linearized model satisfy $\tilde{\mu} = \left(1 - \beta_b \beta_s^{-1}\right) \left(1 - \beta_b \gamma_d\right)^{-1}$, $\tau_h = \left(\beta_s - \tilde{\mu}\left(1 - \gamma_d\right)\Theta - \beta_b\right)$ and $\beta = \xi \beta_b + (1 - \xi)\beta_s$. The parameter $\tau_h$ is a housing tax applied to ensure that the steady state is efficient.

[24]The loss function in equation (52) corresponds to the social welfare function derived as a second order approximation to household welfare. Ferrero et al. (2018) show that the weights on policy objectives in the loss functions are related to the deep parameters of the model as follows: $\lambda_\pi = \frac{\varepsilon}{\gamma}$; $\lambda_c = \frac{\xi(1-\xi)\sigma(1+\sigma+\varphi)}{(1+\varphi)(\sigma+\varphi)}$; $\lambda_h = \frac{\sigma_h \xi(1-\xi)}{\sigma+\varphi}$. Ferrero et al. (2018) also consider cases in which macroprudential policy instruments are used alongside monetary policy to maximize social welfare.

the model, (1), to give a model incorporating the optimal policy response of the form:

$$H_D^F \mathbb{E}_t x_{t+1} + H_D^C x_t + H_D^B x_{t-1} = \Psi_D z_t$$

3. Form the forward shock loading matrix, $F_D$, using equation (3): $F_D = -\left(H_D^C + H_D^F B_D\right)^{-1} H_D^F$.
4. Use the AIM solution representation, (2), to compute the effects of the news shock, namely:

$$x_t = B_D x_{t-1} + \mathbb{E}_t \sum_{i=0}^{\infty} F_D^i \Phi_D z_{t+i}$$

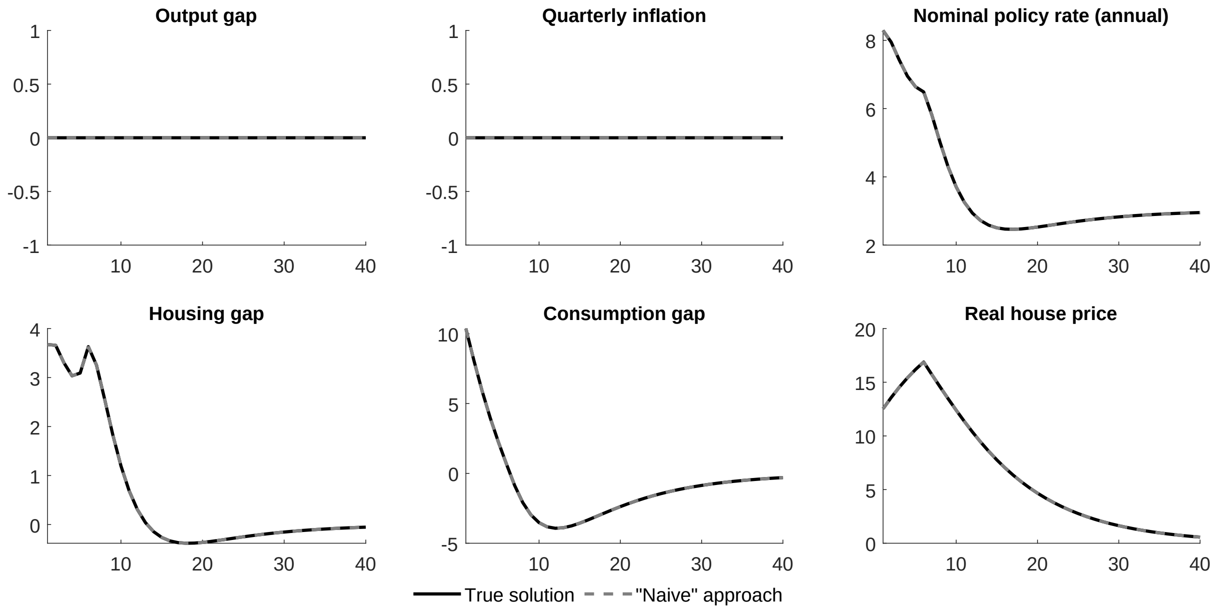


**Figure 2:** Housing demand news shock: “flexible inflation targeting” loss function

Figure 2 shows a housing preference news shock. In quarter 0, the model is at steady state. In quarter 1, it is revealed that a positive housing preference disturbance will arrive in quarter 6 (that is, $\varepsilon_6^h > 0$). The solid black lines depict the results from the solution algorithm described in Section 4.1 and the dashed gray lines show the results from applying the ‘naive’ approach described above.

In this case, the naive approach delivers the same outcome as the correct solution. The reason is that the first order condition for optimal policy delivers a targeting rule for monetary policy that is entirely static. In particular, as shown by Ferrero et al. (2018), the targeting rule under this policy configuration is given by:

$$\pi_t = -\frac{1}{\varepsilon} x_t \tag{53}$$

which is the familiar targeting rule from the textbook New Keynesian model under optimal discretion (for example, Woodford, 2003). Intuitively, this criterion emerges because, in the absence of a lower bound constraint on the policy rate, the monetary policymaker’s current decisions have no effect on the ability of future policymakers to set optimal policy.

More formally, given the aggregate demand structure of the model, the output gap can be treated as the policy instrument. As a result, the Phillips curve is the only binding constraint on the policy problem so that the values of endogenous state variables (for example, debt) do not restrict the set of allocations that can be achieved by the current policymaker.

The results in Figure 2 demonstrate that a housing demand news shock leads to an immediate rise in the real house price, which continues to increase until the shock is realized. Thereafter, real house prices decline back towards steady state. The targeting criterion (53) is delivered by complete stabilization of the output gap and inflation. The path for the policy instrument therefore follows the path of the natural real interest rate, which itself is determined by the path of the consumption gap.[25]

The prospect of higher future house prices relaxes the loan-to-value constraint on borrowers, leading them to increase consumption and their holdings of housing in the near term, financed by higher real debt levels. These patterns move into reverse once the shock has been realized and real house prices subsequently decline, following the autoregressive process (50).

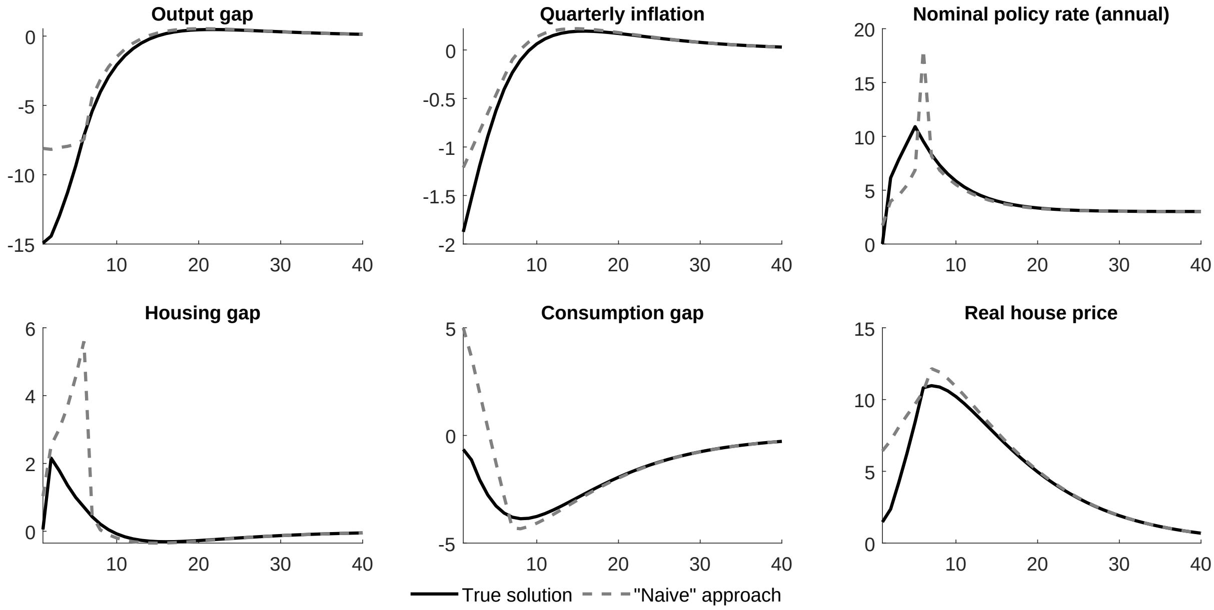


**Figure 3:** Housing demand news shock: "leaning against the wind" loss function

Figure 3 shows the effects of the same shock when the policymaker minimizes the LAW loss function, (52). In this case, the constraints imposed by the full dynamic structure of the model (rather than just the Phillips curve) are relevant. As a result, the true solution is very different to one constructed using the 'naive' approach.

The optimal policy in this case does not stabilize the output gap and inflation. Rather, the optimal policy generates a large negative output gap and (hence) a sizable undershoot of inflation below target. This is achieved by a persistent increase in the real interest rate. So, relative to the flexible inflation targeting case (Figure 2), leaning against the wind generates a tighter policy stance and smaller fluctuations in the consumption and (to a lesser degree) housing gaps. Tighter policy also moderates the initial increase in the real house price.

The dynamics for the 'naive' solution approach are markedly different for many variables. Even when the broad contours of the responses are similar, the magnitudes are often substantially different (for example, the response of inflation). More generally, the naive solution tends to generate solutions that are more similar to the flexible inflation targeting case, shown in Figure 2. In part this reflects the fact that, unlike the true solution, the naive solution does not properly account for the dynamic responses of the private sector and monetary policymaker.

Note that the naive solution approach *could* be used to generate the correct solution when the policymaker minimizes (52), as long as the news shocks are directly encoded into the structural form of the model, (1). Such an approach would involve adding a set of equations that measure the effect of a shock in the past (see, for example, Harrison, 2015). Doing so ensures that the policymaker and private sector internalize the effects of news shocks on decisions at all horizons.

[25] See Ferrero et al. (2018) for further discussion.

While such an approach can be easily used for a small number of news shocks, it does not generalize to the case when many of these shocks are used as 'anticipated disturbances' in a broader sense. One example of this is when anticipated disturbances are used to construct a baseline forecast containing judgment, demonstrated in the example presented in Section 7.

# 5 Optimal discretion with instrument bounds

The method to impose instrument bounds on the optimal commitment solution described in Section 3 leveraged the fact that the decision rules for optimal policy (the first order conditions) have a time-invariant form.[26] That permitted the model augmented with the first order conditions for optimal policy to be solved as a rational expectations model. The Holden and Paetz (2012) method can then be used to impose instrument bounds by finding the anticipated sequence of values for the Lagrange multipliers on the instrument constraints that implement the constraints.

In the presence of instrument bounds, the first order conditions for optimal discretionary policy do not, in general, have a time-invariant form. This implies that, in general, the Holden and Paetz (2012) approach cannot be used to apply instrument bounds when policy is set under optimal discretion. This is discussed further in Section 5.2 and demonstrated using a simple example in Appendix F.

Incorporating instrument bounds under optimal discretionary policy therefore requires the use of a method that is able to correctly internalize the time-varying nature of optimal policy behavior. One such approach is to cast the policy problem in terms of a finite horizon dynamic programming problem, under the (verifiable) assumption that no instrument constraints bind beyond the horizon under consideration.

This section details an implementation of the method developed in Brendon, Paustian, and Yates (2010, henceforth 'BPY'), extended to include the effects of anticipated disturbances. This approach delivers optimal outcomes in a time-consistent Markov-perfect Stackelberg-Nash equilibrium under the assumption of perfect foresight.

The algorithm works by iterating over indicators of binding constraints. It first solves for a terminal steady state in which the constraints on policy instruments are slack. It then starts from a guess of the path of constraints indicators, which take the value of 1 if a particular constraint is binding in a particular period and 0 if it is slack, in a transition from the current state to the terminal state. It then solves the model backwards using value function iterations from the terminal state, assuming that constraints are binding as suggested by the guess of the indicators. This procedure results in a set of time-varying policy rules for the instruments. The algorithm then checks the constraints and the non-negativity of Lagrange multipliers on the inequality constraints for the guess of binding constraints. If the optimality conditions are satisfied, the time-varying policy rules represent the solution. If they are not, the guess for the constraint indicators is updated.[27]

## 5.1 Method

Let $H$ denote the horizon over which future disturbances may be anticipated *and* the horizon over which constraints may be binding. That is, no constraints bind beyond horizon $H$.[28]

[26] As discussed in Section 3, while a 'timeless perspective' solution gives rise to truly time invariant solution, the first period solution must be adjusted when computing a fully optimal solution.

[27] For example, a candidate solution will be inadmissible if either of the following are true: (a) the candidate solution for one or more of the instruments violates the bounds placed on it; (b) the candidate solution for one or more of the Lagrange multipliers on the instrument constraints violate the non-negativity constraint.

[28] This assumption is unlikely to be restrictive because it is unlikely that there would be many (if any) relevant applications. In any case, it would be reasonably straightforward to extend the derivation to allow for return to a steady state in which one or more instrument bound constraints is binding.

As in previous sections, the model is (5), repeated here for convenience:

$$\widetilde{H}^F_{\tilde{x}}\mathbb{E}_t\widetilde{x}_{t+1} + \widetilde{H}^C_{\tilde{x}}\widetilde{x}_t + \widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} + \widetilde{H}^F_r\mathbb{E}_t r_{t+1} + \widetilde{H}^C_r r_t = \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t$$

The policymaker minimizes the same discounted sum of current and future losses, (6), but also subject to bound constraints on the instruments (54):

$$\begin{aligned}&\min_{x_t, r_t} \mathcal{L}_t = \mathbb{E}_t\sum_{i=0}^{\infty}\beta^i\left\{(\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + (r_{t+i})' Q (r_{t+i})\right\}\\ &s.t.\ Sr_t \geq b\end{aligned} \tag{54}$$

where $S$ is an $n_\mu \times n_r$ coefficient matrix, and $b$ is an $n_\mu \times 1$ vector of constants that characterize the instrument bounds.

The solution to be found will be of the following form:

$$\widetilde{x}_t = B_{\tilde{x}\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Xi_{s,\tilde{x}\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\tilde{x},t} \tag{55}$$

$$r_t = B_{r\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Xi_{s,r\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{r,t} \tag{56}$$

$$\mu_t = B_{\mu\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Xi_{s,\mu\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\mu,t} \tag{57}$$

where $\mu$ denotes the vector of Lagrange multipliers attached to the bound constraints, the vectors $\gamma_{\tilde{x},t}$, $\gamma_{r,t}$ and $\gamma_{\mu,t}$ measure the impact of the constraints (should they be binding contemporaneously or in the future) on the endogenous variables, instruments and Lagrange multipliers in period $t$ and where $\Xi_{s,\tilde{x}\tilde{z},t}$, $\Xi_{s,r\tilde{z},t}$ and $\Xi_{s,\mu\tilde{z},t}$ are loadings measuring the impact of anticipated disturbances $s$ periods ahead on the endogenous variables, instruments and Lagrange multipliers in period $t$.

Appendix D solves the optimal policy problem by backward induction starting from period $H$. That process demonstrates the following results.

The constraint internalized by the policymaker in a generic period $t$ is given by:

$$\widetilde{x}_t = \Theta_t^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^F_{\tilde{x}}\sum_{s=0}^{H-t-1}\Xi_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+1+s} - \widetilde{H}^F_r\sum_{s=0}^{H-t-1}\Xi_{s,r\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r r_t - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},t+1} - \widetilde{H}^F_r\gamma_{r,t+1}\end{pmatrix} \tag{58}$$

where:

$$\Theta_t = \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}}B_{\tilde{x}\tilde{x},t+1} + \widetilde{H}^F_r B_{r\tilde{x},t+1} \tag{59}$$

Appendix D demonstrates that the first order condition for the instrument can be expressed as:

$$\begin{aligned}Qr_t - S'\mu_t &= \zeta_t\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^F_{\tilde{x}}\sum_{s=0}^{H-t-1}\Xi_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+1+s} - \widetilde{H}^F_r\sum_{s=0}^{H-t-1}\Xi_{s,r\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r r_t - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},t+1} - \widetilde{H}^F_r\gamma_{r,t+1}\end{pmatrix}\\ &+ \left(\Theta_t^{-1}\widetilde{H}^C_r\right)'\beta\left(\sum_{s=1}^{H-t}V_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+s} + V_{\tilde{x}\gamma,t+1}\right)\end{aligned}$$

where:

$$\zeta_t = \left(\Theta_t^{-1}\widetilde{H}^C_r\right)'(W + \beta V_{\tilde{x}\tilde{x},t+1})\,\Theta_t^{-1} \tag{60}$$

The first order condition can be written more compactly as:

$$\Delta_{r,t}r_t - S'\mu_t = \Delta_{\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Delta_{\tilde{z}_s,t}\widetilde{z}_{t+s} + \Delta_{c,t} \tag{61}$$

where:

$$\begin{aligned}
\Delta_{r,t} &= Q + \zeta_t \widetilde{H}_r^C && (62)\\
\Delta_{\tilde{x},t} &= -\zeta_t H_{\tilde{x}}^B\\
\Delta_{\tilde{z}_0,t} &= \zeta_t \widetilde{\Psi}_{\tilde{z}}\\
\Delta_{\tilde{z}_s,t} &= \left(\Theta_t^{-1}\widetilde{H}_r^C\right)'\beta V_{s,\tilde{x}\tilde{z},t+1} - \zeta_t\left(\widetilde{H}_{\tilde{x}}^F \Xi_{s-1,\tilde{x}\tilde{z},t+1} + \widetilde{H}_r^F \Xi_{s-1,r\tilde{z},t+1}\right)\\
\Delta_{c,t} &= \left(\Theta_t^{-1}\widetilde{H}_r^C\right)'\beta V_{\tilde{x}\gamma,t+1} - \zeta_t\left(\widetilde{H}_{\tilde{x}}^F \gamma_{\tilde{x},t+1} + \widetilde{H}_r^F \gamma_{r,t+1}\right) && (63)
\end{aligned}$$

Appendix E demonstrates that the derivatives of future losses satisfy:

$$\begin{aligned}
V_{\tilde{x}\tilde{x},t+1} &= \left(B_{\tilde{x}\tilde{x},t+1}\right)'\left(W + \beta V_{\tilde{x}\tilde{x},t+2}\right)B_{\tilde{x}\tilde{x},t+1} + \left(B_{r\tilde{x},t+1}\right)' Q B_{r\tilde{x},t+1} && (64)\\
V_{s,\tilde{x}\tilde{z},t+1} &= \left(B_{\tilde{x}\tilde{x},t+1}\right)'\left(W + \beta V_{\tilde{x}\tilde{x},t+2}\right)\Xi_{s-1,\tilde{x}\tilde{z},t+1} + \left(B_{r\tilde{x},t+1}\right)' Q \Xi_{s-1,r\tilde{z},t+1}\\
&\quad + \left(B_{\tilde{x}\tilde{x},t+1}\right)'\beta V_{s-1,\tilde{x}\tilde{z},t+2} && (65)\\
V_{\tilde{x}\gamma,t+1} &= \left(B_{\tilde{x}\tilde{x},t+1}\right)'\left(W + \beta V_{\tilde{x}\tilde{x},t+2}\right)\gamma_{\tilde{x},t+1} + \left(B_{r\tilde{x},t+1}\right)' Q\gamma_{r,t+1} + \left(B_{\tilde{x}\tilde{x},t+1}\right)'\beta V_{\tilde{x}\gamma,t+2} && (66)
\end{aligned}$$

The equilibrium conditions for the period $t$ instruments and Lagrange multipliers can therefore be written as:

$$\begin{bmatrix} \Delta_{r,t} & -S' \\ \mathbb{J}_t S & \mathbb{I} - \mathbb{J}_t \end{bmatrix}\begin{bmatrix} r_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} \Delta_{\tilde{x},t} \\ 0 \end{bmatrix}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\begin{bmatrix} \Delta_{\tilde{z}_s,t} \\ 0 \end{bmatrix}\widetilde{z}_{t+s} + \begin{bmatrix} \Delta_{c,t} \\ \mathbb{J}_t b \end{bmatrix} \tag{67}$$

where $\mathbb{J}_t$ is an $n_\mu \times n_\mu$ diagonal matrix indicating which of the constraints is binding in period $t$. This system jointly determines the solution for the instruments and the Lagrange multipliers:

$$r_t = B_{r\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Xi_{s,r\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{r,t} \tag{68}$$

$$\mu_t = B_{\mu\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t}\Xi_{s,\mu\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\mu,t} \tag{69}$$

where:

$$\begin{aligned}
B_{r\tilde{x},t} &= \Gamma_{rr,t}\Delta_{\tilde{x},t} && (70)\\
\Xi_{s,r\tilde{z},t} &= \Gamma_{rr,t}\Delta_{\tilde{z}_s,t}\\
\gamma_{r,t} &= \Gamma_{rr,t}\Delta_{c,t} + \Gamma_{r\mu,t}\mathbb{J}_t b\\
B_{\mu\tilde{x},t} &= \Gamma_{\mu r,t}\Delta_{\tilde{x},t}\\
\Xi_{s,\mu\tilde{z},t} &= \Gamma_{\mu r,t}\Delta_{\tilde{z}_s,t}\\
\gamma_{\mu,t} &= \Gamma_{\mu r,t}\Delta_{c,t} + \Gamma_{\mu\mu,t}\mathbb{J}_t b && (71)
\end{aligned}$$

where $\Gamma_{rr,t}$, $\Gamma_{r\mu,t}$, $\Gamma_{\mu r,t}$ and $\Gamma_{\mu\mu,t}$ are the upper-left, upper-right, lower-left and lower-right blocks of

$\begin{bmatrix} \Delta_{r,t} & -S' \\ \mathbb{J}_t S & \mathbb{I}-\mathbb{J}_t \end{bmatrix}^{-1}$ respectively, defined as:

$$\begin{aligned}
\Gamma_{rr,t} &= \Delta_{r,t}^{-1} - \Delta_{r,t}^{-1} S' \Gamma_{\mu\mu,t} \mathbb{J}_t S \Delta_{r,t}^{-1} \qquad (72)\\
\Gamma_{r\mu,t} &= \Delta_{r,t}^{-1} S' \Gamma_{\mu\mu,t}\\
\Gamma_{\mu r,t} &= -\Gamma_{\mu\mu,t} \mathbb{J}_t S \Delta_{r,t}^{-1}\\
\Gamma_{\mu\mu,t} &= \left(\mathbb{I} - \mathbb{J}_t + \mathbb{J}_t S \Delta_{r,t}^{-1} S'\right)^{-1} \qquad (73)
\end{aligned}$$

Substituting the law of motion for the instruments into the constraint in equation (58) gives:

$$\widetilde{x}_t = B_{\tilde{x}\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \Xi_{s,\tilde{x}\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\tilde{x},t} \qquad (74)$$

where:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^{B} + \widetilde{H}_r^{C} B_{r\tilde{x},t}\right) \qquad (75)\\
\Xi_{0,\tilde{x}\tilde{z},t} &= \Theta_t^{-1}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}_r^{C}\Xi_{0,r\tilde{z},t}\right)\\
\Xi_{s,\tilde{x}\tilde{z},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^{F}\Xi_{s-1,\tilde{x}\tilde{z},t+1} + \widetilde{H}_r^{F}\Xi_{s-1,r\tilde{z},t+1} + \widetilde{H}_r^{C}\Xi_{s,r\tilde{z},t}\right)\\
\gamma_{\tilde{x},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},t+1} + \widetilde{H}_r^{F}\gamma_{r,t+1} + \widetilde{H}_r^{C}\gamma_{r,t}\right) \qquad (76)
\end{aligned}$$

### 5.1.1 The algorithm

The following algorithm can be used to compute a perfect-foresight transition given a set of anticipated disturbances, $\{\widetilde{z}_t\}_{t=1}^{H}$, from an arbitrary initial condition, $\widetilde{x}_0$, to a steady state regime (pre-computed using the method in Appendix A) in which none of the instrument constraints is binding.

0. Form a guess for the indicators, $\{\mathbb{J}_t\}_{t=1}^{H}$, that describe which of the constraints is binding in each period of the transition.

1. For each period, $t$, starting in period $H$ (using the period $H+1$ terminal conditions outlined at the end of Appendix D.1) and working backwards to period 1, compute the following:

   (a) Compute $\Theta_t$ and $\zeta_t$ using equations (59) and (60).

   (b) Compute $\Delta_{r,t}$, $\Delta_{\tilde{x},t}$, $\Delta_{c,t}$, $\Delta_{\tilde{z}_0,t}$ and, if $t < H$, $\{\Delta_{\tilde{z}_s,t}\}_{s=1}^{H-t}$ using equations (62)-(63).

   (c) Compute $\Gamma_{rr,t}$, $\Gamma_{r\mu,t}$, $\Gamma_{\mu r,t}$ and $\Gamma_{\mu\mu,t}$ using the formulae in equations (72)-(73).

   (d) Compute and store $B_{r\tilde{x},t}$, $\{\Xi_{s,r\tilde{z},t}\}_{s=0}^{H-t}$, $\gamma_{r,t}$, $B_{\mu\tilde{x},t}$, $\{\Xi_{s,\mu\tilde{z},t}\}_{s=0}^{H-t}$ and $\gamma_{\mu,t}$ using equations (70)-(71) and $B_{\tilde{x}\tilde{x},t}$, $\{\Xi_{s,r\tilde{z},t}\}_{s=0}^{H-t}$ and $\gamma_{r,t}$ using equations (75)-(76).

   (e) Compute $V_{\tilde{x}\tilde{x},t}$, $V_{\tilde{x}\gamma,t}$ and $\{V_{s,\tilde{x}\tilde{z},t}\}_{s=1}^{H-t}$ using equations (64)-(66).

2. Compute $\{\widetilde{x}_t\}_{t=1}^{H}$, $\{r_t\}_{t=1}^{H}$ and $\{\mu_t\}_{t=1}^{H}$ starting in period $t=1$ and working forwards to period $H$. If $Sr_t - b \geq 0$ and $\mu_t \geq 0$ for all $t = 1 \ldots H$, then an equilibrium has been found. If not, then the guess $\{\mathbb{J}_t\}_{t=1}^{H}$ does not constitute an equilibrium, either because one or more of the constraints is violated in one or more periods, or because one or more of the Lagrange multipliers is negative (indicating that a constraint is assumed to be binding when it should not be). In that case, update the guess $\{\mathbb{J}_t\}_{t=1}^{H}$ and go back to step 1.

## 5.2 Discussion

As in the method presented in Section 4, a key innovation is the inclusion of anticipated disturbances in the solution. Importantly, the impact of anticipated disturbances is summarized with single horizon and time-varying impact matrices (denoted $\Xi$). It would be possible to factor out the forward loading and shock impact matrices in the same way as for the unconstrained problem considered in Section 4, but there is little value in the additional algebra necessary to do that. The time variation in the solution matrices that is due to variation in whether and the extent to which the instrument bound constraints are binding (either in period $t$ or in expectation in the future) also implies that both the forward and impact loadings will vary over time. So, unlike the method in Section 4, there is no generic expression for the forward loading matrices as a function of the expectation horizon only.

These observations also imply that the case of optimal discretionary policy with instrument bounds cannot, in general, be solved using a straightforward application of the Holden and Paetz (2012) method. It should be emphasized that this result does not represent a criticism of Holden and Paetz (2012): their method is designed for particular cases and they do not claim that optimal discretionary policy is among them.

Appendix F considers this issue in detail and uses a simple two-period example to demonstrate several key results. First, it is possible to replicate the solution presented in this section using the Holden and Paetz (2012) method under some special conditions. These include cases in which (a) there is a single constraint on a single instrument that binds only in the first period of a simulation/projection or (b) the first order condition for optimal discretionary policy is entirely static. Case (a) is clearly a very restrictive one. The 'static' first order conditions in case (b) tend to occur in models in which there are no (payoff relevant) endogenous state variables, which rules out most realistic applications.

Taken together, these results are used to demonstrate that a straightforward application of the Holden and Paetz (2012) will not in general deliver the correct solution because the effects of *future* constraints on policy are not properly accounted for. While Appendix F considers this analytically for a simple example, some intuition for this result can be found by considering the solutions under commitment and discretion from previous sections. For the case of commitment, the time-invariant nature of the first order conditions for optimal policy gives a representation of equilibrium as a function of future multipliers on the instrument constraints, equation (14). A similar representation could be derived in the case of optimal discretion. However, the solution under optimal discretion reveals that the multipliers on the instrument constraints are endogenous to the solution, as shown by equation (69).

Intuitively, a policymaker can seek to moderate the impact of future instrument constraints on the discounted sum of future losses via the optimal decisions they make. A standard application of the Holden and Paetz (2012), which treats the future path of multipliers as exogenous (and isomorphic to anticipated shocks), would not take that dependency into account.

Appendix F provides some further evidence that it may be possible to extend the Holden and Paetz (2012) method to incorporate the endogeneity of the multipliers on the instrument constraints. However, a solution computed using such a method would be conditional on a particular conjecture for the periods in which the instrument constraints were binding. Such an extension would therefore require the same type of 'guess and verify' procedure as the method presented in Section 5.1 in order to find the equilibrium. As such, the strong computational advantages of the Holden and Paetz (2012) method in the case of commitment solutions do not carry over to the case of optimal discretion.

Indeed, all steps of the algorithm in Section 5.1.1 are straightforward and cheap to compute. They are simply applications of the recursive formulas in the definition of equilibrium (in a finite-horizon dynamic program with known terminal and initial conditions).[29] However, the algorithm has the potential to be

[29]The implementation also allows for unanticipated disturbances to be realized in any period along the transition, facilitating stochastic simulation around a baseline scenario or forecast. This requires repeated (recursive) application of the algorithm above, beginning in period $t = 1$ and ending in period $H$. In each period $t$, the

very inefficient, depending on how many iterations are required to find the equilibrium binding constraint indicator $\{\mathbb{J}_t\}_{t=1}^H$. As noted by Brendon et al. (2010), "The precise sequencing of regimes unfortunately requires some guesswork".[30]

The baseline implementation of the algorithm is based on a heuristic that sequentially adds and then removes guesses that constraints are binding (i.e. unit entries in $\{\mathbb{J}_h\}_{h=1}^H$) until an equilibrium has been found. The choice of which guesses to add/remove is based on the largest violations of the constraints on the policy instruments and non-negativity constraints on the associated multipliers. At each iteration, a binding indicator guess is added for the largest violation of the instrument constraint. In the event that there are no such violations, a binding indicator guess is removed for the largest $\mu$ non-negativity violation.[31]

There is relatively little research on the questions of existence and uniqueness of equilibrium in the environment studied here. Armenter (2018) has noted that multiplicity of Markov-perfect equilibria may be widespread in even very simple New Keynesian models in which monetary policy is constrained by a lower bound. From a practical perspective, such a result would imply that the method used to update the sequence of constraint indicators $\{\mathbb{J}_t\}_{t=1}^H$ could influence the (non-unique) equilibrium on which the algorithm settles. Alternatively, there is a possibility that equilibria in which policy instruments are subject to occasionally binding constraints may not exist in discrete time models (see Appendix B in Boneva et al., 2018, for a discussion).

### 5.3 Example

The example revisits the QE experiment of Section 3.3, utilizing the same model, parameterization and shock scenario. However, in this case optimal policy is time consistent (rather than optimal commitment).

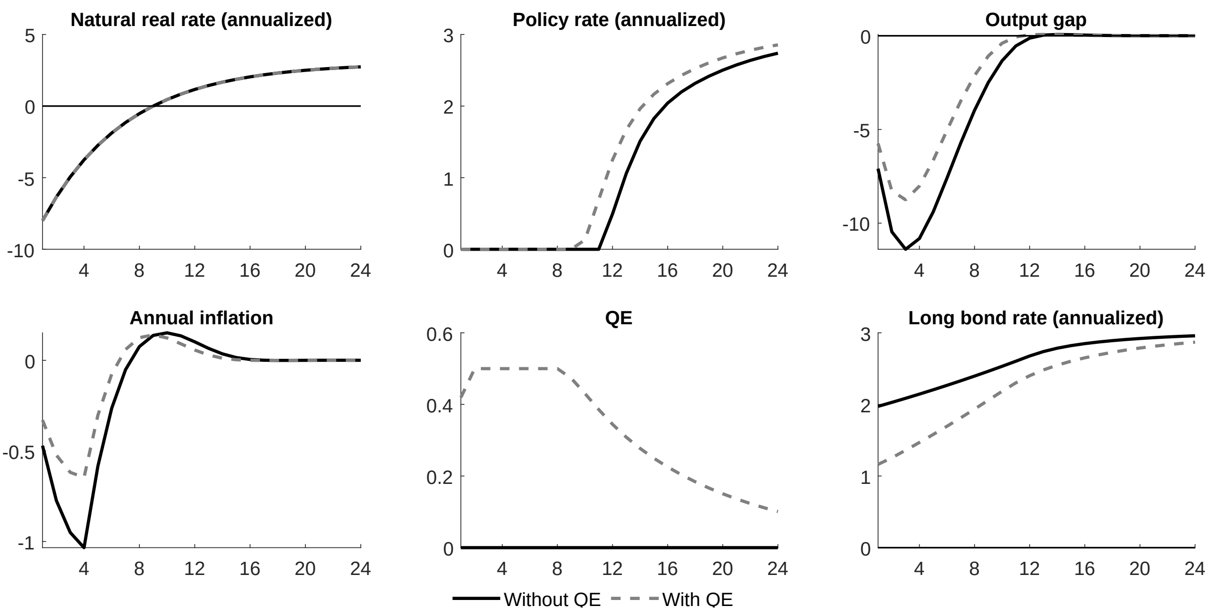


**Figure 4:** Recessionary shock under optimal discretion with instrument bounds

outcomes in periods $t+1 \dots H$, $\{\widetilde{x}_{t+s}\}_{s=1}^{H-t}$, $\{r_{t+s}\}_{s=1}^{H-t}$ and $\{\mu_{t+s}\}_{s=1}^{H-t}$ constitute a forecast conditional on the information known (i.e. the set of anticipated disturbances and the unanticipated shocks realized to date) up to that point.

[30] The 'regimes' in Brendon et al. (2010) correspond to the sequence of indicators $\{\mathbb{J}_t\}_{t=1}^H$.

[31] This approach to incrementally adjusting the guesses for $\{\mathbb{J}_t\}_{t=1}^H$ has the potential to generate 'cycles' in the guesses: the same sequences of guesses for $\{\mathbb{J}_t\}_{t=1}^H$ are repeated as the iterations proceed. Such cycles could be indicative of equilibrium non-existence. It would of course be possible to implement a more sophisticated search, drawing on the literature on discrete optimization.

Figure 4 shows the results of the same recessionary scenario considered in Section 3.3, in which the natural real interest rate falls to -8% under optimal discretion. As before, the solid black lines show the case in which the policymaker has access only to the short-term policy rate and the dashed gray lines show the case in which QE is also used.

Relative to the results under optimal commitment (Figure 1), the shock leads to a much larger and more persistent recession, regardless of whether or not the policymaker has access to QE. The prolonged recession is sufficient to pull inflation below the target (of 0%). These results reflect the well-known result that time-consistent policy is unable to influence current conditions by credible promises to behave in a particular manner in the future. In the context of the recessionary scenario considered here, policymakers are unable to promise to accommodate a future boom in order to increase inflation expectations and hence reduce real interest rates. As a result, time consistent policy leads to a larger and more prolonged recession and the short-term policy rate is constrained by the zero bound for *longer* than when policy is set under optimal commitment.

QE is a relatively powerful additional instrument under discretion. Compared to the case of optimal commitment considered in Section 3.3, the policymaker is unable to rely on costless promises about future interest-rate policy, which motivates a greater reliance on QE. The optimal QE policy (dashed lines) is to enact very large scale asset purchases, so that the upper bound on QE is reached within 2 quarters. A prolonged period of maximal QE is sufficient to lower long-term interest rates persistently. The macroeconomic effect of QE amounts to several percentage points for the output gap and almost 0.5pp for annual inflation.[32]

# 6 Optimal policy with instrument and non-instrument bounds

Many applications of piecewise linear methods to handle occasionally binding constraints (Guerrieri and Iacoviello, 2015; Holden and Paetz, 2012) examine cases in which these constraints may apply to variables other than the policy instruments. For example, Guerrieri and Iacoviello (2017) estimate a model in which the borrowing constraint of impatient households may become slack, in addition to a lower bound on the short-term nominal interest rate (which is assumed to be set according to a simple rule). In contrast, algorithms designed to consider optimal policy in which instruments may be constrained (for example Brendon et al., 2010) do not allow for constraints on variables other than the policy instruments.

This section considers the case in which non-instrument variables may be subject to occasionally binding constraints in addition to the occasionally binding constraints on policy instruments considered in previous sections. A challenge for creating a general purpose algorithm to handle this case is that constraints on non-instrument variables can take a wide variety of forms. The method presented here is designed to encompass a wide variety of plausible cases. In particular, multiple constraints on non-instrument variables are considered, given the potential importance of such cases in future research.[33]

It is assumed that the non-instrument variables are subject to $N \geq 1$ contemporary slackness conditions of the form:

$$C_i \widetilde{x}_t \geq d_i, \qquad G_i \widetilde{x}_t \geq k_i, \qquad (C_i \widetilde{x}_t - d_i)(G_i \widetilde{x}_t - k_i) = 0, \quad i = 1, \ldots, N \tag{77}$$

The 'baseline' state of the model (described in equation (5)) is assumed to be the case in which all

[32]Note that the overshoot of inflation in Figure 4 reflects the role of the model dynamics (in particular the lag of the output gap in the Phillips curve) rather than the result of a credible promise to engineer a period of above target inflation.

[33]The inclusion of multiple constraints can lead to rich and state-dependent dynamic as demonstrated by Bluwstein et al. (2020) in the case of occasionally binding constraints on leverage and new lending.

constraints are 'inactive'. By convention, the term 'inactive' is used to refer to cases in which

$$C_i \widetilde{x}_t = d_i,$$

and, without loss of generality, it is assumed that:

$$d_i = 0, \quad i = 1, \ldots, N$$

which means that this case satisfies:

$$\begin{bmatrix} C_1 \\ \vdots \\ C_N \end{bmatrix} \widetilde{x}_t = 0 \tag{78}$$

The motivation for labeling these cases as 'inactive' states is that there are many relevant cases in which constraints on non-instrument variables are generally binding, but may occasionally become slack. A common example is the case of a collateral constraint. Many models assume that the collateral constraint binds in the deterministic steady state and for small shocks around that steady state. However, the constraint may become temporarily slack in response to a large enough shock (see Guerrieri and Iacoviello, 2017; Ferrero et al., 2018, for example). As such, it may be unhelpful to label alternative states of the model as those in which non-instrument constraints are 'binding' or 'slack'. Indeed, the very nature of the contemporary slackness conditions implies that one constraint will bind when another is slack. So, for the example of a collateral constraint that binds near to the steady state, the binding collateral constraint is labeled as the 'inactive' state and the case in which the constraint is slack is the 'active' state.

Since there are $N$ non-instrument OBCs that may each be in either of two states (i.e., 'inactive', $C_i \widetilde{x}_t = 0$; or 'active', $G_i \widetilde{x}_t = k_i$) there are a total of $2^N$ distinct states. These distinct states are indexed $j = 1, \ldots, 2^N$, with the 'baseline' state identified as state 1.

The framework outlined above implies that the structural form of the model (5) can be written as follows:

$$\widetilde{H}^F_{\tilde{x}} \mathbb{E}_t \widetilde{x}_{t+1} + \sum_{j=1}^{2^N} \mathcal{I}_{j,t} \widetilde{H}^C_{\tilde{x},<j>} \widetilde{x}_t + \widetilde{H}^B_{\tilde{x}} \widetilde{x}_{t-1} + \widetilde{H}^F_r \mathbb{E}_t r_{t+1} + \widetilde{H}^C_r r_t = \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t + \Psi_\delta \sum_{j=1}^{2^N} \mathcal{I}_{j,t} \delta_{<j>}$$

where $\mathcal{I}_{j,t}$ is a scalar indicator variable taking the value of 1 if the model is in state $j$ in period $t$ and 0 otherwise and the additional disturbances $\delta$ are introduced to enforce the equality constraints $G_i \widetilde{x}_t = k_i$ as described below. Note that the notation '$< j >$' is used to index the 'state' of the model. The matrix $\Psi_\delta$ is an $n_{\tilde{x}} \times N$ matrix that loads on the $N \times 1$ vector $\delta$. So the $i$-th row of $\delta$ records a scalar value (1 or 0) corresponding to whether or not the $i$-th constraint is active in period $t$.[34] Similarly the $i$-th column of $\Psi_\delta$ is zero, except for the row corresponding to the equation of the structural form that implements the constraint $G_i \widetilde{x}_t = k_i$: that element of $\Psi_\delta$ is equal to $k_i$.

The validity of the representation above relies on several assumptions about the implementation of the non-instrument constraints. The main assumption is that the equations corresponding to the constraints (78) are a subset of the $\widetilde{H}^C_{\tilde{x}}$ matrix in the baseline state (ie $\widetilde{H}^C_{\tilde{x},<1>}$). Those constraints also imply that $\delta_{<1>} = 0$. Importantly, it is assumed that these equations only feature entries in the $\widetilde{H}^C_{\tilde{x}}$ matrix (and not in $\widetilde{H}^F_{\tilde{x}}$, $\widetilde{H}^B_{\tilde{x}}$ or $\widetilde{\Psi}_{\tilde{z}}$). This is not a restrictive assumption because it is possible to include additional equations within the model structure as identities that define the variables to be constrained.

Given this observation, a second assumption is that each contemporary slackness condition is written

[34] As noted previously, there are $2^N$ possible values of $\delta$ corresponding to whether or not each of the $N$ constraints are active or inactive.

as a constraint on a single element of $\widetilde{x}$ (so it is identified with a particular variable). In practice, this means that the $G_i$ matrices are $1 \times n_{\tilde{x}}$ vectors of zeros with a single unit entry in the column corresponding to the variable that must satisfy the constraint. While not strictly necessary for the implementation of the method (and it is not imposed in the derivation below), this assumption simplifies the algorithmic implementation of the method.

Given these assumptions, the structural form of the model can be written more compactly as:

$$\widetilde{H}_{\tilde{x}}^F \widetilde{x}_{t+1} + \widetilde{H}_{\tilde{x},t}^C \widetilde{x}_t + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{t-1} + \widetilde{H}_r^F r_{t+1} + \widetilde{H}_r^C r_t = \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t + \Psi_\delta \delta_t \tag{79}$$

where the time-varying coefficient matrices, $\widetilde{H}_{\tilde{x},t}^C$ and $\delta_t$ depend on the set of OBCs that are active in period $t$. Given the focus on a perfect foresight solution the expectation operator is omitted (as in previous sections).

The next subsections consider the cases of optimal commitment and discretion when the structural model equations are given by (79). Similar to the approach described in Section 5, the solution assumes that anticipated disturbances ($\widetilde{z}$) are zero beyond some horizon, $H$. Moreover, it is assumed that the model is permanently in the baseline state from period $H+1$ onwards.[35]

## 6.1 Optimal commitment

The time-varying nature of the structural model equations implies that the solution does not take a time-invariant form as considered in Section 3. Instead, it is necessary to solve backwards from period $H+1$, in a similar manner to the methods used for optimal discretion in Sections 4 and 5. In common with those methods, the solution approach involves making a guess for the instrument and non-instrument OBCs that bind in each period and then verifying whether that guess is valid. Since the solution is computed by backward induction, only the timeless perspective case is considered here.[36]

Given the structural form (79), the first order conditions of the optimal policy problem can be written as:

$$0 = Qr_t - \left(\widetilde{H}_r^C\right)' \lambda_t - \beta^{-1} \left(\widetilde{H}_r^F\right)' \lambda_{t-1} - S' \mu_t \tag{80}$$

$$0 = W\widetilde{x}_t - \left(\widetilde{H}_{\tilde{x},t}^C\right)' \lambda_t - \beta^{-1} \left(\widetilde{H}_{\tilde{x}}^F\right)' \lambda_{t-1} - \beta \left(\widetilde{H}_{\tilde{x}}^B\right)' \mathbb{E}_t \lambda_{t+1} \tag{81}$$

$$0 = \widetilde{H}_{\tilde{x}}^F \mathbb{E}_t \widetilde{x}_{t+1} + \widetilde{H}_{\tilde{x},t}^C \widetilde{x}_t + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{t-1} + \widetilde{H}_r^F \mathbb{E}_t r_{t+1} + \widetilde{H}_r^C r_t - \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \Psi_\delta \delta_t \tag{82}$$

Following the logic of Section 3, it is helpful to rewrite the model in terms of an expanded vector of variables and shocks:

$$y_t \equiv \begin{bmatrix} \widetilde{x}_t \\ r_t \\ \lambda_t \end{bmatrix}, \quad \widehat{z}_t \equiv \begin{bmatrix} \widetilde{z}_t \\ \delta_t \end{bmatrix}$$

With these expanded vectors, the structural form of the model is given by:

$$H_y^F y_{t+1} + H_{y,t}^C y_t + H_y^B y_{t-1} = \widehat{\Psi}_{\hat{z}} \widehat{z}_t + \Psi_\mu \mu_t \tag{83}$$

[35] Provided a solution exists, this is not a restrictive assumption. $H$ can be increased to be sufficiently large for the model to revert to the baseline state and a projection from period $H+1$ onwards can be used to verify that none of the occasionally binding constraints are subsequently violated.

[36] As in Section 3, consideration of the fully optimal solution requires a straightforward adjustment to the first order conditions in the first period.

where the matrices $H_y^F$ and $H_y^B$ are reported in Section 3 and where:

$$H_{y,t}^C = \begin{bmatrix} 0 & Q & -\left(\widetilde{H}_r^C\right)' \\ W & 0 & -\left(\widetilde{H}_{\widetilde{x},t}^C\right)' \\ \widetilde{H}_{\widetilde{x},t}^C & \widetilde{H}_r^C & 0 \end{bmatrix}$$

$$\widehat{\Psi}_{\hat{z}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \tilde{\Psi}_{\tilde{z}} & \Psi_\delta \end{bmatrix}$$

$$\Psi_\mu = \begin{bmatrix} S' \\ 0 \\ 0 \end{bmatrix}$$

Appendix G shows that the solution is given by:

$$y_t = B_{y,t} y_{t-1} + \sum_{h=0}^{H-t} F_{t,h} \widehat{\Phi}_{t+h} \widehat{z}_{t+h} + \sum_{h=0}^{H-t} F_{t,h} \Phi_{\mu,t+h} \mu_{t+h} \tag{84}$$

where

$$\begin{aligned} B_{y,t} &= -\left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} H_y^B \\ \widehat{\Phi}_t &= \left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,t} &= \left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} \Psi_\mu \end{aligned}$$

and the $B_{y,t}$ matrix recursions start from

$$B_{y,H+1} = B_y$$

where $B_y$ is the rational expectations solution from equation (14).

For $h > 1$, the $F$ matrices are given by:

$$F_{t,h} = \Upsilon_t F_{t+1,h-1}$$

with:

$$\begin{aligned} & F_{t,0} = \mathbb{I} \\ & \Upsilon_t \equiv -\left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} H_y^F \end{aligned}$$

which implies $F_{t,1} = \Upsilon_t$.

As in the method in Section 3, computing the results of a simulation or projection of the model that incorporates the effects of the instrument constraints uses $\mu_t$ $(t = 1, \ldots, H)$ to impose the constraints, if necessary. The difference is that the simulation or projection now also uses $\delta_t$ $(t = 1, \ldots, H)$ to ensure that non-instrument constraints are also and simultaneously satisfied.

Again following the method of Section 3, the first step is to construct a 'baseline simulation' in which the instrument constraints are assumed not to bind: assuming $\mu_t = 0$, $t = 1, \ldots, H$. From a given initial condition $\{\widetilde{x}_0, \lambda_0\}$, a realization of the anticipated disturbances $\{\widetilde{z}_t\}_{t=1}^H$ and an assumption about the non-instrument constraints that are active in each period (and hence $\left\{H_{y,t}^C, \delta_t\right\}_{t=1}^H$), the baseline simulation for $\{\widetilde{x}_t\}_{t=1}^H$ is computed using (84) (with $\mu_t = 0$, $t = 1, \ldots, H$).

The baseline simulation can be checked to determine whether it violates the assumption that the instrument constraints never bind. This amounts to checking whether $Sr_t > b,\ t = 1, \ldots, H$. If any of these assumptions is violated in the baseline simulation, the instrument constraints are enforced using a variant of the method in Section 3.

As in Section 3, the occasionally binding constraints are implemented by computing the effects of on the 'target' variables, $\tau_t$ defined in equation (15). From (84) the effects of the $\mu$ shocks $\{\mu_t\}_{t=1}^{H}$ on the endogenous variables in period 1 of the simulation is given by:

$$\widehat{y}_1 = \sum_{h=0}^{H-1} F_{1,h}\Phi_{\mu,1+h}\mu_{1+h} \tag{85}$$

and the effects on the policy instruments are given by $\widehat{\tau}_1 = S_\tau \widehat{y}_1$.

As in Section 3, a recursive scheme for building a matrix that maps the effects $\mu_t,\ t = 1, \ldots, H$ to the instruments in each period can be developed. The first (block) row of the matrix can be found by expanding (85):

$$\widehat{\tau}_1 = S_\tau \underbrace{\begin{bmatrix} \Phi_{\mu,1} & \ldots & F_{1,t-1}\Phi_{\mu,t} & \ldots & F_{1,H-1}\Phi_{\mu,H} \end{bmatrix}}_{\equiv \omega_1} \underbrace{\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_t \\ \vdots \\ \mu_H \end{bmatrix}}_{\equiv \mathcal{D}}$$

so that $\omega_t \mathcal{D}$ denotes the effects of current and future multipliers on (all) variables ($y$) in period $t$.

The weights on current and future policy shocks at horizon $h$, $\omega_h$, are given by:

$$\begin{aligned}
\omega_2 &= \begin{bmatrix} 0 & \Phi_{\mu,2} & \ldots & F_{2,t-1}\Phi_{\mu,t} & \ldots & F_{2,H-2}\Phi_{\mu,H} \end{bmatrix} \\
&\ldots \\
\omega_t &= \begin{bmatrix} 0 & 0 & \ldots & \Phi_{\mu,t} & \ldots & F_{t,H-t}\Phi_{\mu,H} \end{bmatrix} \\
&\ldots \\
\omega_H &= \begin{bmatrix} 0 & 0 & \ldots & 0 & \ldots & \Phi_{\mu,H} \end{bmatrix}
\end{aligned}$$

This implies that the effects of past and future shadow shocks at horizon $h$ is given by:

$$\mathcal{R}_t = B_{y,t}\mathcal{R}_{y-1} + \omega_t, \qquad t = 1, \ldots, H \tag{86}$$

with $\mathcal{R}_0 = 0$.

As in Section 3, these coefficients can be used to build a matrix mapping the $\mu$ shocks to the target variables as:

$$\mathcal{T} = \mathcal{M}\mathcal{D} \tag{87}$$

where

$$\mathcal{T} = \begin{bmatrix} \widehat{\tau}_1 \\ \vdots \\ \widehat{\tau}_h \\ \vdots \\ \widehat{\tau}_H \end{bmatrix}, \qquad \mathcal{M} = \begin{bmatrix} S_\tau\mathcal{R}_1 \\ \vdots \\ S_\tau\mathcal{R}_h \\ \vdots \\ S_\tau\mathcal{R}_H \end{bmatrix} \tag{88}$$

Equation (87) is isomorphic to equation (20) in Section 3 and so the same quadratic programming approach can be used to find the sequence of $\{\mu_t\}_{t=1}^H$ required to implement the occasionally binding constraints on the policy instruments.

Collecting together the steps described above gives the following algorithm:

0. Form an initial guess for the equilibrium set of non-policy constraints that are active in each period (encoded in, $\{\mathcal{I}_{j,t}\}_{t=1}^H$, $j = 1, \ldots, 2^N$). Set $\{\delta_t\}_{t=1}^H$ consistent with this guess.
1. For each period, $t$, starting in period $H$ (using the period $H+1$ terminal conditions outlined in Appendix G) and working backwards to period 1, compute $B_{y,t}, \widehat{\Phi}_t, \Phi_{\mu,t}$ and $\{F_{t,t+h}\}_{h=0}^{H-t}$ using the formula above.
2. Compute $\{y_t\}_{t=1}^H$ using (84) starting in period $t = 1$ and working forwards to period $H$ and assuming $\mu_t = 0, t = 1, \ldots, H$.
3. Check the constraints:
   (a) If $Sr_t - b \geq 0$ is violated in any period $t$, form the matrices $\mathcal{T}$ and $\mathcal{M}$ in equation (88) and solve the quadratic programming problem in (21)–(23) for the multiplier sequence $\{\mu_t\}_{t=1}^H$. Recompute the projection using (84) and $\{\mu_t\}_{t=1}^H$. Otherwise go directly to step 3b.
   (b) If the contemporary slackness conditions (77) are satisfied for all $i = 1, \ldots, N$ and for all $t = 1, \ldots, H$, the solution has been found. Otherwise, update the guess $\{\mathcal{I}_{j,t}\}_{t=1}^H$, $j = 1, \ldots, 2^N$ and go back to step 1.

## 6.2 Optimal discretion

The algorithm for optimal discretion subject to instrument constraints presented in Section 5 is easily adapted to handle the type of non-instrument constraints considered in this section. That is because the algorithm embeds the time-varying nature of the policy problem (in particular, the effects of the occasionally binding instrument constraints on the derivatives of future losses with respect to current allocations).

As described earlier, the method for incorporating non-instrument constraints encapsulates those constraints in the form of a time-varying matrix $\widetilde{H}_{\tilde{x},t}^C$. Inspection of the derivation described in Section 5 reveals that the time-variation in the structural equations can be captured by updating equation (59) to

$$\Theta_t = \widetilde{H}_{\tilde{x},t}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x},t+1} + \widetilde{H}_r^F B_{r\tilde{x},t+1} \tag{89}$$

where $\widetilde{H}_{\tilde{x},t}^C$ depends on the constraints that are active in period $t$, as in the derivation in Section 6.1.

The structure of the model must also be extended to include anticipated disturbances, $\delta$ used to impose the effects of the occasionally binding constraints on the constrained variables. To preserve the notation in Section 5 this can be done by defining the vector of non-policy shocks to be:

$$\tilde{z}_t \equiv \begin{bmatrix} \tilde{z}_t^\dagger \\ \delta_t \end{bmatrix} \tag{90}$$

where $\tilde{z}^\dagger$ are the 'fundamental' non-policy shocks in the structural model, (5). Similarly,

$$\widetilde{\Psi}_{\tilde{z}} = \begin{bmatrix} \widetilde{\Psi}_{\tilde{z}}^\dagger & \Psi_\delta \end{bmatrix} \tag{91}$$

where $\widetilde{\Psi}_{\tilde{z}}^\dagger$ represents the loadings on the 'fundamental' non-policy shocks in the original model.

The solution algorithm is therefore given by:

0. Extend the model (5) to include the additional disturbances, $\delta$ using equations (90) and (91). Form an initial guess for the equilibrium set of non-policy constraints that are active in each period (encoded in, $\{\mathcal{I}_{j,t}\}_{t=1}^H$, $j = 1, \ldots, 2^N$). Set $\{\delta_t\}_{t=1}^H$ consistent with this guess.

1. Solve the optimal discretion problem, subject to instrument bounds, using the algorithm presented in Section 5.1.1, but using equation (89) in place of (59) to compute $\Theta_t$.
2. Check the constraints. If the contemporary slackness conditions (77) are satisfied for all $i = 1, \ldots, N$ and for all $t = 1, \ldots, H$, the solution has been found. Otherwise, update the guess $\{\mathcal{I}_{j,t}\}_{t=1}^{H}$, $j = 1, \ldots, 2^N$ and go back to step 1.

## 6.3 Discussion

The methods described in the previous sub-section are straightforward extensions of those presented in earlier sections. As such, the same general issues apply. In particular, the previously discussed issues associated with finding a solution using a 'guess and verify' approach will be particularly relevant for the discretionary equilibrium described in Section 6.2. One benefit of the commitment solution described in Section 6.1 is that the quadratic programming approach to finding the sequence of multipliers $\{\mu_t\}_{t=1}^{H}$ is likely to be faster than a simple iterative scheme for updating guesses about the instrument constraints that bind each period.

Importantly, a variant of the Holden and Paetz (2012) method *can* be used for commitment policies, even though the structural equations of the model vary over time on account of the occasionally binding non-policy constraints. This is possible because the changes in the structural equations are perfectly foreseen (conditional on the conjectured sequence of non-policy constraints that bind in each period) and because the Holden and Paetz (2012) is adapted to account for the implications of this time variation for the effects of future instrument constraints on endogenous variables (that is, the $F_{t,h}$ and $\Phi_{\mu,t}$ matrices in equation (84)).

## 6.4 Examples

### 6.4.1 Commitment

This example uses the model of Smets and Wouters (2007, henceforth 'SW'), viewed by many as an exemplar of medium-scale DSGE models estimated using Bayesian methods. It has been the blueprint for models developed for forecasting and policy analysis in many policy institutions. Since the model is well known, the description below is confined to the small modifications required for the experiment.

The starting point is the SW model with parameter values equal to the posterior means reported in Smets and Wouters (2007). Two modifications are made to the model for the purposes of this example.

The first modification raises the persistence of the "risk premium shock" ($\rho_b$) to 0.85 (the posterior mean is 0.22). This creates more protracted responses to the shock, which is necessary to produce meaningful episodes in which occasionally binding constraints are relevant.

The second modification is to replace the baseline specification of investment adjustment costs with an assumption that investment is subject to 'speed limit' constraints, specified below. Removing investment adjustment costs implies that the parameter $\varphi$ is set to zero.

The constraints on investment can be written as:

$$\Delta i_t \leq \varpi^U \tag{92}$$

$$\Delta i_t \geq \varpi^L \tag{93}$$

where $\Delta i_t \equiv i_t - i_{t-1}$ is the change in investment and $\varpi^U > 0 > \varpi^L$ are the upper and lower limits on the rate of change of investment.

Given the absence of investment adjustment costs, these constraints imply that the investment Euler equation (equation (3) in Smets and Wouters (2007)) is replaced by:

$$q_t - \xi_t + \beta\gamma^{1-\sigma}\xi_{t+1} + \zeta_t - \beta\gamma^{1-\sigma}\zeta_{t+1} = 0$$

where $\xi$ and $\zeta$ are the Lagrange multipliers on constraints (92) and (93) respectively.[37]

The contemporary slackness conditions for this particular case are therefore given by:

$$\underbrace{\zeta_t \geq 0}_{C_1\widetilde{x}_t \geq 0}, \qquad \underbrace{\Delta i_t \geq \varpi^L}_{G_1\widetilde{x}_t \geq k_1}, \qquad \underbrace{(\zeta_t - 0)\left(\Delta i_t - \varpi^L\right) = 0}_{(C_1\widetilde{x}_t - 0)(G_1\widetilde{x}_t - k_1) = 0} \tag{94}$$

$$\underbrace{\xi_t \geq 0}_{C_2\widetilde{x}_t \geq 0}, \qquad \underbrace{-\Delta i_t \geq -\varpi^U}_{G_2\widetilde{x}_t \geq k_2}, \qquad \underbrace{(\xi_t - 0)\left(-\Delta i_t - \left(-\varpi^U\right)\right) = 0}_{(C_2\widetilde{x}_t - 0)(G_2\widetilde{x}_t - k_2) = 0} \tag{95}$$

The 'speed limits', $\varpi^U$ and $\varpi^L$, are set to limit changes in investment to roughly 4 percentage points above and below steady-state investment growth. While this limit is set arbitrarily (for the purposes of the simulation), it nevertheless allows for relatively large swings in investment, given that the model is calibrated on a quarterly basis.[38]

The policymaker uses the nominal interest rate, $r$, to minimize the loss function

$$\mathcal{L}_t = \sum_{i=0}^{\infty} \beta^i \left(\pi_{t+i}^2 + \lambda_x x_{t+i}^2\right)$$

where $x$ is the flexible price measure of the output gap. Following the example in Section 6.4.2, the weight on the output gap in the loss function is set to $\lambda_x = 0.25$.

Loss minimization is subject to a lower bound on the policy rate:

$$r_t \geq b$$

where the bound $b$ is set equal to minus 1 times the steady-state nominal interest rate, computed from the posterior mean parameter values reported by Smets and Wouters (2007).

The scenario considered is similar to the one in Section 6.4.2. A large negative shock to the risk premium process ($\varepsilon^b$) is anticipated to arrive in quarter $t = 4$. The scale of the shock is chosen to require a material response of the monetary policy rate such that the lower bound becomes a relevant constraint on the policy response.

Figure 5 plots the results.

In the absence of any occasionally binding constraints (dashed gray lines) the optimal policy path necessitates a reduction in the policy rate below zero when the risk premium shock arrives in quarter 4. As in the example in Section 6.4.2, a path for the policy rate that tracks the evolution of the risk premium process delivers full stabilization of the output gap and inflation.

In the presence of the zero bound on the policy rate (solid gray lines), the policy rate is constrained by the zero bound when the shock arrives in quarter 4. In contrast with the example in Section 6.4.2, a policymaker that is able to commit to future policy actions avoids a deep recession in the short term. Conditional on the lower bound on the policy rate, the optimal policy is to promise a prolonged period of relative loose monetary policy (a low path for the *real* interest rate). This sustains a long-lived output boom and a persistent rise in inflation. The rise in inflation is sufficiently large that the nominal interest

[37] Note that the constraints on investment are specified in the underlying decision problem, so that (92) and (93) represent log-linearized versions of the constraints and $\xi$ and $\zeta$ are transformed versions of the underlying multipliers on those constraints (that is, scaled by the marginal utility of consumption). It is assumed that the flexible price equilibrium is not subject to investment speed limits.

[38] In the updated version of the SW data set used by Harrison (2015), these upper and lower bounds were each violated just twice between 1980 and 2008.

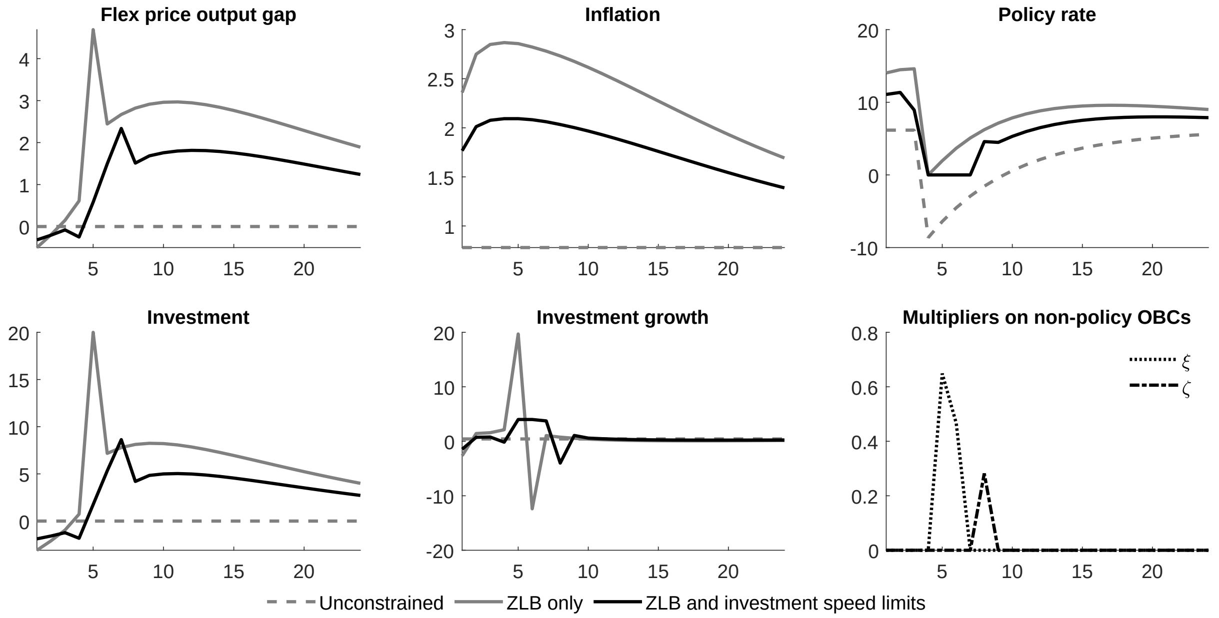


**Figure 5:** Negative risk premium shock scenario under optimal commitment with zero bound and 'speed limit' constraints on investment growth

rate is generally above its steady-state level. Since investment is fully flexible, investment and investment growth follow particularly volatile profiles.

The solid black lines show the case in which the investment 'speed limits', (94) and (95), are imposed alongside the lower bound on the policy rate. The limits on investment growth reduce the amplitude of the investment responses in the quarters around the arrival of the $\varepsilon^b$ shock. The extent to which investment can rise when the risk premium shock arrives is particularly constrained. This limits the degree to which the path of monetary policy can stimulate spending and inflation and the optimal policy path is constrained at the zero bound for several quarters after the shock arrives. Nonetheless, the resulting deviations of output from potential and inflation from target are *smaller* in the presence of additional constraints on the feasible paths for investment. This result echoes that of Section 6.4.2 and highlights the potential for a subtle interplay between constraints on policy instruments and non-policy variables.

The final panel plots the multipliers $\xi$ and $\zeta$ for the simulation in which both the zero bound and investment speed limits are imposed. In equilibrium, the period for which the investment speed limits are binding overlaps with the period during which the monetary policymaker is constrained by the zero lower bound.

#### 6.4.2 Discretion

A simple inertial New Keynesian model is used to explore the implications of downward nominal wage rigidity. Most of the model is standard and resembles the model used in Sections 3.3 and 5.3 (though without a role for QE).

$$x_t - \eta x_{t-1} = \mathbb{E}_t \left( x_{t+1} - \eta x_t \right) - \sigma \left( i_t - \mathbb{E}_t \pi_{t+1} - r_t^* \right) \tag{96}$$

$$\pi_t - \iota \pi_{t-1} = \beta \mathbb{E}_t \left( \pi_{t+1} - \iota \pi_t \right) + \alpha^{-1} \left( 1 - \beta \alpha \right) \left( 1 - \alpha \right) w_t \tag{97}$$

$$r_t^* = \rho r_{t-1}^* + \varepsilon_t \tag{98}$$

The IS curve (96) includes habit formation, with habit parameter $\eta \in [0, 1)$. The Phillips curve (97) includes price indexation, with indexation parameter $\iota \in [0, 1)$. The Phillips curve is written in terms of the real wage $w$ because the focus will be on the case in which nominal wages are sticky downwards. The weight on the real wage (real marginal cost) is a function of the Calvo adjustment parameter $\alpha$.

With fully flexible nominal wages, the real wage will equal the marginal rate of substitution, which is given by:[39]

$$m_t \equiv \left(\psi + \frac{1}{\sigma}\right) x_t - \frac{\eta}{\sigma} x_{t-1}$$

However, suppose that workers will not accept a fall in the absolute level of nominal wages. Under the assumption that the model is log-linearized around a positive (quarterly) inflation target, $\pi^*$, the following constraint is applied:

$$\pi_t^w \geq -\pi^* \tag{99}$$

where

$$\pi_t^w \equiv w_t - w_{t-1} + \pi_t$$

is the quarterly rate of wage inflation.

To implement the non-negativity constraint on wage inflation, the gap between the real wage and the marginal rate of substitution is defined by:

$$\widetilde{w}_t \equiv w_t - m_t$$

The 'baseline state' of the model corresponds to the case in which $\widetilde{w}_t = 0$ and the occasionally binding constraint (the lower bound on nominal wage inflation) is 'inactive'. The contemporary slackness condition for this particular case is therefore given by:

$$\underbrace{\widetilde{w}_t \geq 0}_{C_1 \widetilde{x}_t \geq 0}, \qquad \underbrace{\pi_t^w \geq -\pi^*}_{G_1 \widetilde{x}_t \geq k_1}, \qquad \underbrace{(\widetilde{w}_t - 0)\left(\pi_t^w - (-\pi^*)\right) = 0}_{(C_1 \widetilde{x}_t - 0)(G_1 \widetilde{x}_t - k_1) = 0} \tag{100}$$

The policymaker uses the nominal interest rate, $i$, to minimize the loss function

$$\mathcal{L}_t = \sum_{i=0}^{\infty} \beta^i \left(\pi_{t+i}^2 + \lambda_x x_{t+i}^2\right)$$

subject to a lower bound on the policy rate:

$$i_t \geq \ln \frac{\beta}{\pi^*}$$

The parameter values used in the experiment are shown in Table 3. For the most part, these values are taken from the model in used in Sections 3.3 and 5.3. The value of $\beta$ is higher, to ensure that the steady state real rate is lower (at around 2% annualized). This implies that the steady-state *nominal* interest rate is around 4% (annualized), given the assumption of a 2% annual inflation target. The persistence of the shock to the natural real interest rate is higher, to increase the scale of the stabilization problem (and hence ensure that the lower bound on the policy instrument and nominal wage inflation are encountered in a meaningful manner).

The simulation is an *anticipated* (or 'news') shock to the natural rate of interest. Given the calibration of the model, each period of time is interpreted as a quarter (of a year). The value of $\varepsilon_t$ in quarter $t = 4$

[39] This can be derived for a period utility function defined over consumption, $C$, and hours worked $N$ of the form $U(C, N) = \frac{\left(C_t - \eta \bar{C}_{t-1}\right)^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} + \frac{\chi}{1+\psi} N_t^{1+\psi}$, where habit formation is defined in terms of aggregate consumption $\bar{C}$. The result imposes market clearing, $y_t = c_t$; the production function $y_t = n_t$ (both expressed in first order approximations using log-deviations) and the fact that the preference shock does not affect potential output. So the output gap satisfies $x_t = y_t$.

| Parameter | Description | Value |
|---|---|---|
| $\beta$ | Household discount factor | 0.995 |
| $\alpha$ | Probability of not adjusting price | 0.8725 |
| $\psi$ | Inverse Frisch elasticity | 0.11 |
| $\sigma$ | Elasticity of intertemporal substitution | 1 |
| $\iota$ | Degree of inflation indexation | 0.7 |
| $\rho$ | Autocorrelation of natural real interest rate | 0.9 |
| $\eta$ | Habit formation parameter | 0.8 |
| $\lambda_x$ | Loss function weight on output gap | 0.25 |
| $100\pi^*$ | Inflation target (quarterly, per cent) | 0.5 |

**Table 3:** Parameter values for inertial New Keynesian model

is chosen so that the (annualized) natural rate of interest falls from its steady-state value of 4% to -4% in quarter 4 of the simulation. Thereafter, it recovers according to the autoregressive process (98).

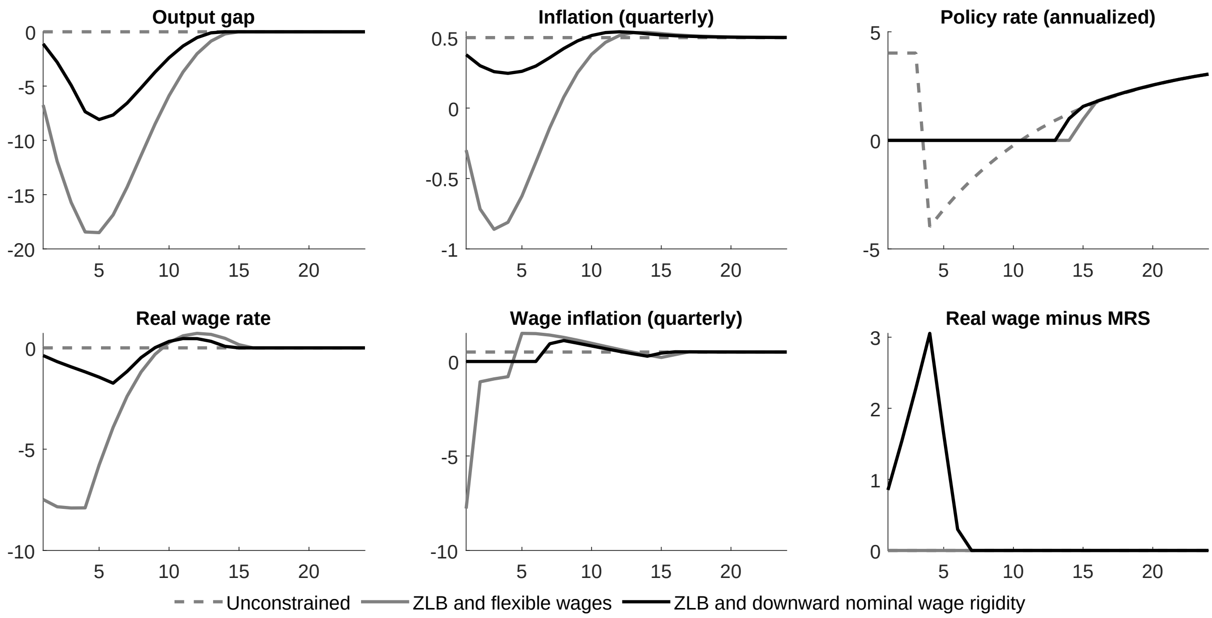


**Figure 6:** Recessionary scenario under optimal discretion with zero bound and downward nominal wage rigidity

Figure 6 shows the results of the simulation.

In the absence of any occasionally binding constraints (dashed gray lines) the optimal policy path necessitates a reduction in the policy rate below zero in quarter 4, before tracking the recovery of the natural real interest rate back to steady state. This ensures full stabilization of output, price and wage inflation and the real wage.

In the presence of the zero bound on the policy rate (solid gray lines), the policy rate cannot track the evolution of $r^*$ when it falls in quarter 4. Agents recognize that the policy rate will be too high in the future: monetary policy will be too tight. This expectation reduces aggregate demand and inflation immediately, even though the natural real interest rate does not fall until quarter 4. Indeed, the fall in demand and inflation is enough to push the policy rate to the zero bound immediately. This implies that monetary policy is too tight for a prolonged period, generating a substantial recession and negative quarterly inflation rates. The large fall in output generates substantial movements in the marginal rate of substitution. Since this simulation assumes fully flexible wages, the real wage falls markedly and wage inflation is substantially negative for several quarters. The policy rate stays at the lower bound for several quarters after the natural real interest rate has risen above zero, reflecting the continued weakness of the output gap and inflation due to the inertial dynamics in the model.

The solid black lines show the case in which downward nominal wage rigidity, equation (99), is imposed alongside the lower bound on the policy rate. In this case, the initial dramatic fall in nominal wage inflation is mitigated, which dampens the fall in the real wage, marginal cost and inflation. The fact that monetary policy is initially constrained by the zero bound implies that the tightening in monetary policy is also moderated, resulting in a smaller fall in the output gap. In equilibrium, the gap between the real wage and the marginal rate of substitution ($\widetilde{w}$, plotted in the bottom right panel) is somewhat smaller than would be implied from a comparison of the solid black and solid gray lines.[40] Overall the presence of downward nominal wage rigidity generates a smaller recession and a slightly faster recovery (the policy rate lifts off from the lower bound earlier than in the flexible wage).

These results demonstrate that the interplay between occasionally binding constraints may be subtle. At first glance, the assumption that nominal wages could not adjust downwards in response to a recessionary shock could be thought to exacerbate the scale of a recession generated by the zero lower bound constraint on monetary policy. However, in equilibrium, the additional nominal friction prevented a large scale decline in costs, prices and inflation, thus mitigating the rise in real interest rates associated with the zero bound recession. This experiment therefore displays the so-called 'paradox of flexibility' associated with New Keynesian models.[41]

# 7 Large-scale model application

The examples in earlier sections have used relatively small-scale models, for ease of exposition. For some of those examples, it would in principle be possible to solve the models using projection methods. This section demonstrates the applicability of the toolkit to large-scale macroeconomic models of the type used in policy institutions. For models of this size, projection methods are not (yet) a feasible alternative to approximate solutions such as the piecewise linear methods presented in this paper.

The example uses a version of the FRB/US model routinely used for forecast and policy analysis by staff at the Federal Reserve Board. The model and baseline forecast are taken from Haberis, Harrison, and Waldron (2019, henceforth 'HHW'), which contains a more detailed description of both components.

The FRB/US model is a large-scale model, developed by economists at the Federal Reserve Board.[42] A linearized version of the model contained in the Macroeconomic Model Data Base (MMB) developed by Wieland et al. (2012) is used.[43] Similar versions of this model have been used for optimal policy simulations supporting the FOMC's communications. In particular, the experiments are similar to those reported by Yellen (2012). Even by the standards of workhorse models in use at policy institutions, FRB/US is large: the version used here has over 350 endogenous variables and more than 60 shocks.[44]

The first step in the exercise is to build a baseline projection. This is done following HHW, who construct a forecast based on the FOMC's December 2012 'Survey of Economic Projections' (SEP). The baseline forecast is constructed by selecting a sequence of anticipated disturbances $\{\widetilde{z}_{T+h}\}_{h=1}^{H}$ for forecast periods $h = 1, \ldots, H$ to deliver projections for a subset of endogenous variables, $\{\widetilde{x}_{T+h}\}_{h=0}^{H}$ that match the median SEP forecasts.[45] This highlights the importance of the anticipated disturbances in analysis

[40] This reflects the endogenous response of the output gap, which determines the response of the marginal rate of substitution, $m_t$.

[41] For further analysis and discussion of the paradox of flexibility see, for example: Eggertsson and Krugman (2012); Kiley (2016); Bhattarai et al. (2018); Billi and Galí (2020).

[42] Brayton and Tinsley (1996) provide a description of the first version of the model.

[43] Two versions are provided in the MMB. The rational expectations variant is used to allow for the expectational effects of optimal policies.

[44] So $n_{\tilde{x}} \approx 350$ and $n_{\tilde{z}} \approx 60$.

[45] The 'inversion algorithm' in the MAPS toolkit, described in Burgess et al. (2013, Appendix C) is used for this step.

of this type. As Svensson and Tetlow (2005) argue:

> Projections and monetary policy decisions cannot rely on models and simple observable data alone. All models are drastic simplifications of the economy, and data give a very imperfect view of the state of the economy. Therefore, judgmental adjustments in both the use of models and the interpretation of their results – adjustments due to information, knowledge, and views outside the scope of any particular model – are a necessary and essential component in modern monetary policy.

Following Svensson and Tetlow (2005), the anticipated disturbances $\{\widetilde{z}_{T+h}\}_{h=1}^{H}$, which they call 'deviations', encapsulate non-model information:

> ...the deviation represents the difference between the model outcomes and the actual realizations of data and includes all extra-model explanations of the actual data. Below, the central bank's judgment will be represented as the central bank's projections of the future deviations. This allows us to incorporate the fact that a considerable amount of judgment is always applied to assumptions and projections.

The exercise constructs optimal policy projections, under both commitment and discretion. The loss function follows Yellen (2012):

$$\mathcal{L}_{T+1} = \sum_{h=0}^{\infty} \beta^h \left[ (\pi_{T+h+1} - \pi^*)^2 + (u_{T+h+1} - u^*)^2 + (r_{T+h+1} - r_{T+h})^2 \right] \quad (101)$$

which applies equal weights to squared deviations of annual inflation ($\pi$) from target ($\pi^*$), unemployment ($u$) from the natural rate ($u^*$) and changes in the federal funds rate ($r$). The inflation target is set to 2% and the natural rate of unemployment in the version of FRB/US used here is 5.5%. The discount factor is set to $\beta = 0.9925$.

The federal funds rate is assumed to be subject to a lower bound of 12.5 basis points. This is encoded as a bound on $r$ of the form:

$$r_{T+h} \geq elb$$

where $elb$ is computed relative to the implied steady-state federal funds rate (assumed to be 4%, based on the long-run SEP projections).

Figure 7 shows the baseline projection (solid black line) and the projections under optimal commitment (solid gray line) and optimal discretion (dashed gray line). The dashed vertical line denotes the end of the historical data (i.e., period $T$).

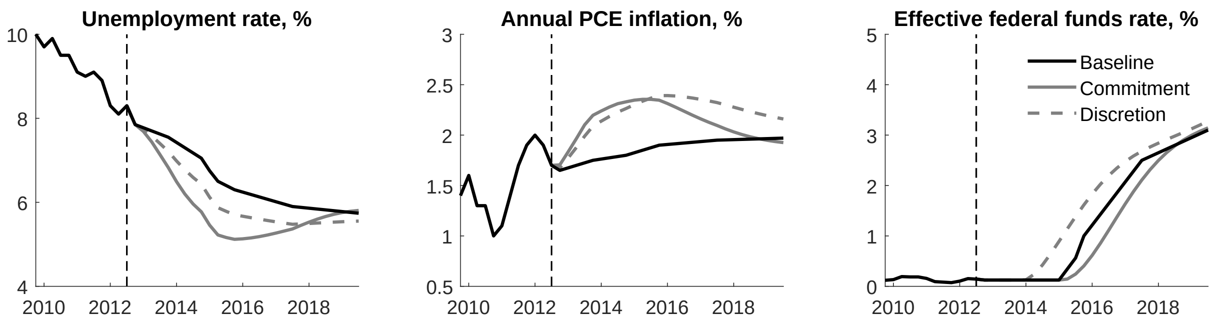


**Figure 7:** Optimal policy projections using FRB/US

The baseline forecast is one in which the unemployment rate gradually falls to around 6% and annual PCE inflation rises gradually to the target of 2%. This is supported by a path for the federal funds rate that stays at the lower bound until 2015, before gradually rising back towards its steady state level.

The projection under optimal commitment (solid gray line) embodies the familiar 'lower for longer' policy prescription. The federal funds rate is held at the lower bound for slightly longer than the baseline forecast and rises more gradually thereafter. This generates sufficient stimulus to reduce unemployment more rapidly. As a result, inflation temporarily overshoots the 2% target.

The projection under optimal discretion (dashed gray line) looks somewhat different. The path for unemployment slightly below the baseline projection, but inflation overshoots the target more persistently. The federal funds rate lifts off somewhat earlier than in the baseline projection, rising more slowly. These properties suggest that the policymaker faces a trade-off between high unemployment and high inflation, that is more difficult to manage when it does not have access to a commitment technology. Under optimal discretion, the policymaker is unable to commit to lifting off later and subsequently raising rates more quickly.

An implication of this result is that the baseline projection may implicitly embody a commitment mechanism that allows for a credible promise to liftoff later than a time-consistent policy would allow.[46] Indeed, these projections were accompanied by an FOMC statement embodying 'threshold-based forward guidance' over the likely path of the federal funds rate (see Haberis et al., 2019). Nonetheless, economic performance is better under optimal discretion than the baseline projection. To examine this, the root mean loss, defined as:

$$\mathcal{M}_T \equiv \left( \frac{1}{H} \sum_{h=0}^{H} \beta^h \left[ \left(\pi_{T+h+1} - \pi^*\right)^2 + \left(u_{T+h+1} - u^*\right)^2 + \left(r_{T+h+1} - r_{T+h}\right)^2 \right] \right)^{\frac{1}{2}}$$

is computed for each projection. The baseline root mean loss is 0.62. Under optimal discretion the mean-square loss falls to 0.52, which is itself higher than the loss of 0.46 achieved under optimal commitment.

# 8 Conclusion

This paper presents a set of piecewise-linear solution algorithms to solve linear rational expectations models in the presence of occasionally binding constraints on policy instruments and non-policy variables. Optimal policy may be computed under the assumption of commitment or discretion.

Importantly, the methods allow for the presence of 'anticipated disturbances' to the model equations. Following Svensson and Tetlow (2005), these anticipated disturbances can be interpreted as capturing judgment and other 'off-model' information that informs the forecasts produced by many policy institutions. Incorporating these disturbances therefore makes it possible to conduct optimal policy analysis with occasionally binding constraints for projections and scenarios produced using judgment. As such, the methods presented here are likely to be of particular use in policy institutions that apply large-scale macroeconomic models to scenarios and forecasts that incorporate judgment and other non-model-based information.

[46] Recall that the baseline projection does not come from the model itself – it is based on the median projections of FOMC members.

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# A The Dennis (2007) algorithm

This appendix, presents a variant of the Dennis (2007) algorithm cast in the same notation used throughout the paper.

For completeness, the statement of the policymaker's problem is restated here. The policymaker minimizes the following quadratic loss function, subject to the constraint imposed by the structure of the economy and taking optimal future policy as given:

$$\begin{aligned}\min_{\widetilde{x}_t, r_t} \mathcal{L}_t &= \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \left\{ (\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + (r_{t+i})' Q (r_{t+i}) \right\} \\ &= (\widetilde{x}_t)' W (\widetilde{x}_t) + (r_t)' Q (r_t) + \beta \mathbb{E}_t \mathcal{L}_{t+1} \end{aligned} \tag{A.1}$$

Given the linearity of the problem (and the assumption that no lags of the instruments appear in the structural equations), the solution that shall be sought and verified below is as follows:

$$\widetilde{x}_t = B_{\tilde{x}\tilde{x}} \widetilde{x}_{t-1} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_t \tag{A.2}$$

$$r_t = B_{r\tilde{x}} \widetilde{x}_{t-1} + \Phi_{r\tilde{z}} \widetilde{z}_t \tag{A.3}$$

which has the same Markovian form as standard RE solutions in linear (or linearized) models.

## A.1 Characterizing the solution

The policymaker's problem is to minimize the loss in equation (38) subject to the constraint imposed by the structure of the economy in equation (5) and taking optimal behavior on the part of future policymakers as given. The optimal behavior of future policymakers can be embedded directly in the constraint imposed by the model by substituting out expectations in the partitioned structural equations (5) using the proposed solution in equations (A.2)-(A.3) and then re-arranging the result to get:

$$\widetilde{x}_t = \Theta^{-1} \left( \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \widetilde{H}^B_{\tilde{x}} \widetilde{x}_{t-1} - \widetilde{H}^C_r r_t \right) \tag{A.4}$$

where:

$$\Theta = \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}} B_{\tilde{x}\tilde{x}} + \widetilde{H}^F_r B_{r\tilde{x}} \tag{A.5}$$

The policymaker's problem can then be represented as a Lagrangean:

$$\min_{\widetilde{x}_t, r_t} \left\{ (\widetilde{x}_t)' W (\widetilde{x}_t) + (r_t)' Q (r_t) + \beta \mathbb{E}_t \mathcal{L}_{t+1} - 2\lambda_t' \left( \widetilde{x}_t - \Theta^{-1} \left( \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \widetilde{H}^B_{\tilde{x}} \widetilde{x}_{t-1} - \widetilde{H}^C_r r_t \right) \right) \right\} \tag{A.6}$$

The first-order conditions are:

$$r_t: \quad 2 (r_t)' Q + \beta \frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial r_t} - 2\lambda_t' \Theta^{-1} \widetilde{H}^C_r = 0 \tag{A.7}$$

$$\widetilde{x}_t: \quad 2 (\widetilde{x}_t)' W + \beta \frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial \widetilde{x}_t} - 2\lambda_t' = 0 \tag{A.8}$$

$$\lambda_t: \quad \widetilde{x}_t - \Theta^{-1} \left( \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \widetilde{H}^B_{\tilde{x}} \widetilde{x}_{t-1} - \widetilde{H}^C_r r_t \right) = 0 \tag{A.9}$$

Appendix A.2 shows that the expectation for next period's loss can be written as:

$$\mathbb{E}_t \mathcal{L}_{t+1} = (\widetilde{x}_t)' V_{\tilde{x}\tilde{x}} \widetilde{x}_t + V_{cc} \tag{A.10}$$

where:

$$V_{\tilde{x}\tilde{x}} = (B_{\tilde{x}\tilde{x}})' W B_{\tilde{x}\tilde{x}} + (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}} \tag{A.11}$$

$$V_{cc} = \frac{1}{1-\beta} tr \left[ \left( (\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}} + (\Phi_{\tilde{x}\tilde{z}})' P \Phi_{\tilde{x}\tilde{z}} \right) \right] \tag{A.12}$$

and:

$$P = W + \beta (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' P B_{\tilde{x}\tilde{x}} \tag{A.13}$$

Fixed points for $V_{\tilde{x}\tilde{x}}$ and $P$ can be found numerically using a doubling algorithm or by solving the implied Lyapunov equation.

It is straightforward to see from equation (A.10) that $\frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial r_t} = 0$. This arises as a consequence of the assumption that no lags of the policy instruments appear in the model equations. It is also straightforward to see that:

$$\frac{\partial \mathbb{E}_t \mathcal{L}_{t+1}}{\partial \widetilde{x}_t} = 2 (\widetilde{x}_t)' V_{\tilde{x}\tilde{x}} \tag{A.14}$$

This can be substituted into first-order condition for the endogenous variables (A.8) and rearrange the result to form an expression for the Lagrange multiplier, $\lambda_t'$:

$$\lambda_t' = (\widetilde{x}_t)' (W + \beta V_{\tilde{x}\tilde{x}}) \tag{A.15}$$

This in turn can be substituted into the first-order condition for the instruments (A.7) to get:[47]

$$\begin{aligned} (r_t)' Q - (\widetilde{x}_t)' (W + \beta V_{\tilde{x}\tilde{x}}) \Theta^{-1} \widetilde{H}_r^C &= 0 \\ Q r_t - \left( \Theta^{-1} \widetilde{H}_r^C \right)' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_t &= 0 \end{aligned} \tag{A.16}$$

Equation (A.16) is the targeting rule that characterises the optimal discretionary policy, relating the optimal choice for the instruments to the optimal choice for the endogenous variables.[48]

Using the first-order condition for the Lagrange multiplier (i.e. the constraint) in equation (A.9) to substitute out $\widetilde{x}_t$ gives:

$$Q r_t - \zeta \left( \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{t-1} - \widetilde{H}_r^C r_t \right) = 0 \tag{A.17}$$

where:

$$\zeta = \left( \Theta^{-1} \widetilde{H}_r^C \right)' (W + \beta V_{\tilde{x}\tilde{x}}) \Theta^{-1} \tag{A.18}$$

Equation (A.17) can be rearranged to get the following:

$$\Delta_r r_t = \Delta_{\tilde{x}} \widetilde{x}_{t-1} + \Delta_{\tilde{z}} \widetilde{z}_t \tag{A.19}$$

where:

$$\Delta_r = Q + \zeta \widetilde{H}_r^C \tag{A.20}$$

$$\Delta_{\tilde{x}} = -\zeta \widetilde{H}_{\tilde{x}}^B \tag{A.21}$$

$$\Delta_{\tilde{z}} = \zeta \widetilde{\Psi}_{\tilde{z}} \tag{A.22}$$

[47] The second line follows from noting that $W$, $Q$ and $V_{\tilde{x}\tilde{x}}$ are symmetric.

[48] The policymaker's optimal choice for the instrument depends on three effects: (i) a direct effect of the instrument on the contemporaneous period loss, which depends on $Q$; (ii) an indirect effect of the instrument on the contemporaneous period loss via its effect on the endogenous variables, which depends on $W$; (iii) an indirect effect of the instrument on the discounted sum of expected future losses, which depends on $\beta V_{\tilde{x}\tilde{x}}$.

It immediately follows that:

$$B_{r\tilde{x}} = \Delta_r^{-1}\Delta_{\tilde{x}} \tag{A.23}$$

$$\Phi_{r\tilde{z}} = \Delta_r^{-1}\Delta_{\tilde{z}} \tag{A.24}$$

which requires that $\Delta_r$ be invertible. Substituting the law of motion for the instruments back into the constraint (A.4) gives:

$$\widetilde{x}_t = \Theta^{-1}\left(\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r\left(B_{r\tilde{x}}\widetilde{x}_{t-1} + \Phi_{r\tilde{z}}\widetilde{z}_t\right)\right) \tag{A.25}$$

From which it is clear that:

$$B_{\tilde{x}\tilde{x}} \;=\; -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r B_{r\tilde{x}}\right) \tag{A.26}$$

$$\Phi_{\tilde{x}\tilde{z}} \;=\; \Theta^{-1}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}^C_r\Phi_{r\tilde{z}}\right) \tag{A.27}$$

which demonstrates that the solution is as proposed in equations (A.2)-(A.3).

## A.2 Period ahead loss function expansion

The expected value of the period ahead loss function is defined as:

$$\begin{aligned}\mathbb{E}_t\mathcal{L}_{t+1} &= \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\left\{(\widetilde{x}_{t+i})' W(\widetilde{x}_{t+i}) + (r_{t+i})' Q(r_{t+i})\right\} \\ &= \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}(\widetilde{x}_{t+i})' W(\widetilde{x}_{t+i}) + \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}(r_{t+i})' Q(r_{t+i})\end{aligned} \tag{A.28}$$

The first term on the right-hand-side of equation (A.28) can be written as:

$$\begin{aligned}\mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}(\widetilde{x}_{t+i})' W(\widetilde{x}_{t+i}) &= \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\left[\begin{array}{c}\left((B_{\tilde{x}\tilde{x}})^i\widetilde{x}_t + \sum_{j=1}^{i}(B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j}\right)' W \\ \left((B_{\tilde{x}\tilde{x}})^i\widetilde{x}_t + \sum_{j=1}^{i}(B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j}\right)\end{array}\right] \\ &= \mathbb{E}_t(\widetilde{x}_t)'\sum_{i=1}^{\infty}\beta^{i-1}\left((B_{\tilde{x}\tilde{x}})^i\right)' W(B_{\tilde{x}\tilde{x}})^i\widetilde{x}_t \\ &+ \mathbb{E}_t(\widetilde{x}_t)'\sum_{i=1}^{\infty}\beta^{i-1}\left((B_{\tilde{x}\tilde{x}})^i\right)' W\sum_{j=1}^{i}(B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j} \\ &+ \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i}(\widetilde{z}_{t+j})'(\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W(B_{\tilde{x}\tilde{x}})^i\widetilde{x}_t \\ &+ \mathbb{E}_t\sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i}(\widetilde{z}_{t+j})'(\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W\sum_{k=1}^{i}(B_{\tilde{x}\tilde{x}})^{i-k}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+k} \\ &= (\widetilde{x}_t)'\sum_{i=1}^{\infty}\beta^{i-1}\left((B_{\tilde{x}\tilde{x}})^i\right)' W(B_{\tilde{x}\tilde{x}})^i\widetilde{x}_t \\ &+ \sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i}tr\left(\left((B_{\tilde{x}\tilde{x}})^i\right)' W(B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\mathbb{E}_t\widetilde{z}_{t+j}(\widetilde{x}_t)'\right) \\ &+ \sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i}tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W(B_{\tilde{x}\tilde{x}})^i\mathbb{E}_t\widetilde{x}_t(\widetilde{z}_{t+j})'\right)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i} tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W\sum_{k=1}^{i}(B_{\tilde{x}\tilde{x}})^{i-k}\Phi_{\tilde{x}\tilde{z}}\mathbb{E}_t\widetilde{z}_{t+k}\left(\widetilde{z}_{t+j}\right)'\right)\\
&= (\widetilde{x}_t)'\sum_{i=1}^{\infty}\beta^{i-1}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\widetilde{x}_t\\
&+\sum_{i=1}^{\infty}\beta^{i-1}\sum_{j=1}^{i} tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\right)\\
&= (\widetilde{x}_t)'(B_{\tilde{x}\tilde{x}})'\sum_{i=0}^{\infty}\beta^{i}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i} B_{\tilde{x}\tilde{x}}\widetilde{x}_t\\
&+\sum_{i=0}^{\infty}\beta^{i}\sum_{j=0}^{\infty}\beta^{j} tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\Phi_{\tilde{x}\tilde{z}}\right)\\
&= (\widetilde{x}_t)'(B_{\tilde{x}\tilde{x}})'\sum_{i=0}^{\infty}\beta^{i}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i} B_{\tilde{x}\tilde{x}}\widetilde{x}_t\\
&+\frac{1}{1-\beta}\sum_{i=0}^{\infty}\beta^{i} tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\Phi_{\tilde{x}\tilde{z}}\right)\\
&= (\widetilde{x}_t)'(B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}}\widetilde{x}_t+\frac{1}{1-\beta} tr\left((\Phi_{\tilde{x}\tilde{z}})' S\Phi_{\tilde{x}\tilde{z}}\right) \qquad \text{(A.29)}
\end{aligned}
$$

where:

$$
S = W+\beta (B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}} \qquad \text{(A.30)}
$$

for which a fixed point can be found numerically using a doubling algorithm.

The steps in the derivation are as follows: the first equality uses the law of motion for the endogenous variables in equation (A.2) to express the loss in terms of time $t$ endogenous variables; the second equality follows by expanding the first; the third equality uses the fact that each term is a scalar, that the trace of a scalar is that scalar and that terms within a trace operator are commutable; the fourth equality follows from noting that the shocks are *iid* (implying zero covariance between shocks in different time periods) and independent of the endogenous variables, and that the covariance matrix of the shocks $\mathbb{E}_t\widetilde{z}_{t+j}\left(\widetilde{z}_{t+j}\right)' = \mathbb{I}$;[49] the fifth equality follows from rewriting the sums as infinite sums starting from zero; the sixth equality follows from noting that $\sum_{j=0}^{\infty}\beta^j \equiv \frac{1}{1-\beta}$; and the seventh (assuming that the spectral radius of $B_{\tilde{x}\tilde{x}}$ is less than one) from:

$$
\begin{aligned}
S &= \sum_{i=0}^{\infty}\beta^{i}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\\
&= W+\sum_{i=1}^{\infty}\beta^{i}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\\
&= W+\beta (B_{\tilde{x}\tilde{x}})'\left\{\sum_{i=0}^{\infty}\beta^{i}\left((B_{\tilde{x}\tilde{x}})^{i}\right)' W (B_{\tilde{x}\tilde{x}})^{i}\right\} B_{\tilde{x}\tilde{x}}\\
&= W+\beta (B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}}
\end{aligned}
$$

[49] This assumption reflects the normalization convention in the MAPS toolkit on which the algorithms are built. In the Dennis (2007) derivation the result is $\mathbb{E}_t\widetilde{z}_{t+j}\left(\widetilde{z}_{t+j}\right)' = \Omega$. A full set of steps for the covariance matrix of the shocks is as follows: $\mathbb{E}_t\left(\widetilde{z}_{t+j}\right)'(\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j} \equiv$ $\mathbb{E}_t tr\left(\left(\widetilde{z}_{t+j}\right)'(\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j}\right) \equiv$ $\mathbb{E}_t tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j}\left(\widetilde{z}_{t+j}\right)'\right) \equiv tr\left(\mathbb{E}_t\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{t+j}\left(\widetilde{z}_{t+j}\right)'\right)\right) \equiv$ $tr\left(\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\mathbb{E}_t\widetilde{z}_{t+j}\left(\widetilde{z}_{t+j}\right)'\right)\right) \equiv tr\left((\Phi_{\tilde{x}\tilde{z}})'\left((B_{\tilde{x}\tilde{x}})^{i-j}\right)' W (B_{\tilde{x}\tilde{x}})^{i-j}\Phi_{\tilde{x}\tilde{z}}\Omega\right)$

The steps above can be repeated for the second term on the right-hand-side of equation (A.28) and use the law of motion for the instruments characterized in equation (A.3) to write:

$$
\begin{aligned}
\mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} (r_{t+i})' Q (r_{t+i}) &= \mathbb{E}_t \left(B_{r\tilde{x}} \widetilde{x}_t + \Phi_{r\tilde{z}} \widetilde{z}_{t+1}\right)' Q \left(B_{r\tilde{x}} \widetilde{x}_t + \Phi_{r\tilde{z}} \widetilde{z}_{t+1}\right) \\
&+ \mathbb{E}_t \sum_{i=2}^{\infty} \beta^{i-1} \left\{ \begin{array}{c} \left( \begin{array}{c} B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i-1} \widetilde{x}_t + \Phi_{r\tilde{z}} \widetilde{z}_{t+i} \\ +B_{r\tilde{x}} \sum_{j=1}^{i-1} (B_{\tilde{x}\tilde{x}})^{i-1-j} \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{t+j} \end{array} \right)' Q \\ \left( \begin{array}{c} B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i-1} \widetilde{x}_t + \Phi_{r\tilde{z}} \widetilde{z}_{t+i} \\ +B_{r\tilde{x}} \sum_{j=1}^{i-1} (B_{\tilde{x}\tilde{x}})^{i-1-j} \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{t+j} \end{array} \right) \end{array} \right\} \\
&= (\widetilde{x}_t)' \sum_{i=1}^{\infty} \beta^{i-1} \left((B_{\tilde{x}\tilde{x}})^{i-1}\right)' (B_{r\tilde{x}})' Q B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i-1} \widetilde{x}_t \\
&+ \mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} (\widetilde{z}_{t+i})' (\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}} \widetilde{z}_{t+i} \\
&+ \mathbb{E}_t \sum_{i=2}^{\infty} \beta^{i-1} \sum_{j=1}^{i-1} \left\{ \begin{array}{c} (\widetilde{z}_{t+j})' (\Phi_{\tilde{x}\tilde{z}})' \left((B_{\tilde{x}\tilde{x}})^{i-1-j}\right)' (B_{r\tilde{x}})' Q \\ B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i-1-j} \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{t+j} \end{array} \right\} \\
&= (\widetilde{x}_t)' \sum_{i=0}^{\infty} \beta^{i} \left((B_{\tilde{x}\tilde{x}})^{i}\right)' (B_{r\tilde{x}})' Q B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i} \widetilde{x}_t + \frac{1}{1-\beta} tr\left((\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}}\right) \\
&+ \frac{\beta}{1-\beta} \sum_{i=0}^{\infty} \beta^{i} tr\left((\Phi_{\tilde{x}\tilde{z}})' \left((B_{\tilde{x}\tilde{x}})^{i}\right)' (B_{r\tilde{x}})' Q B_{r\tilde{x}} (B_{\tilde{x}\tilde{x}})^{i} \Phi_{\tilde{x}\tilde{z}}\right)
\end{aligned}
$$

$$
\mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} (r_{t+i})' Q (r_{t+i}) = (\widetilde{x}_t)' R \widetilde{x}_t + \frac{1}{1-\beta} tr\left((\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}}\right) + \frac{\beta}{1-\beta} tr\left((\Phi_{\tilde{x}\tilde{z}})' R \Phi_{\tilde{x}\tilde{z}}\right) \tag{A.31}
$$

which follows identical (or near identical) steps to those described in the derivation of equation (A.29) and where:

$$
R = (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' R B_{\tilde{x}\tilde{x}} \tag{A.32}
$$

which can also be found using a doubling algorithm.

Finally, the results can bed substituted in equations (A.29) and (A.31) into equation (A.28) to get:

$$
\begin{aligned}
\mathbb{E}_t \mathcal{L}_{t+1} &= \mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} (\widetilde{x}_{t+i})' W (\widetilde{x}_{t+i}) + \mathbb{E}_t \sum_{i=1}^{\infty} \beta^{i-1} (r_{t+i})' Q (r_{t+i}) \\
&= (\widetilde{x}_t)' (B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}} \widetilde{x}_t + \frac{1}{1-\beta} tr\left((\Phi_{\tilde{x}\tilde{z}})' S \Phi_{\tilde{x}\tilde{z}}\right) + (\widetilde{x}_t)' R \widetilde{x}_t + \frac{1}{1-\beta} tr\left((\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}}\right) \\
&+ \frac{\beta}{1-\beta} tr\left((\Phi_{\tilde{x}\tilde{z}})' R \Phi_{\tilde{x}\tilde{z}}\right) \\
&= (\widetilde{x}_t)' \left((B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}} + R\right) \widetilde{x}_t + \frac{1}{1-\beta} tr\left[\left((\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}} + (\Phi_{\tilde{x}\tilde{z}})' P \Phi_{\tilde{x}\tilde{z}}\right)\right] \\
&= (\widetilde{x}_t)' V_{\tilde{x}\tilde{x}} \widetilde{x}_t + V_{cc}
\end{aligned} \tag{A.33}
$$

where:

$$
\begin{aligned}
V_{\tilde{x}\tilde{x}} &= (B_{\tilde{x}\tilde{x}})' S B_{\tilde{x}\tilde{x}} + R \\
&= (B_{\tilde{x}\tilde{x}})' W B_{\tilde{x}\tilde{x}} + (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}} && \text{(A.34)} \\
V_{cc} &= \frac{1}{1-\beta} tr\left[\left((\Phi_{r\tilde{z}})' Q \Phi_{r\tilde{z}} + (\Phi_{\tilde{x}\tilde{z}})' P \Phi_{\tilde{x}\tilde{z}}\right)\right] && \text{(A.35)}
\end{aligned}
$$

with:

$$\begin{aligned} P &= S + \beta R \\ &= W + \beta \left(B_{r\tilde{x}}\right)' Q B_{r\tilde{x}} + \beta \left(B_{\tilde{x}\tilde{x}}\right)' P B_{\tilde{x}\tilde{x}} \end{aligned} \tag{A.36}$$

Note that the definitions for $P$, $S$ and $R$ are identical to those in Dennis (2007). The coefficient $(V_{\tilde{x}\tilde{x}})$ in the quadratic for $\widetilde{x}$ is different (Dennis (2007) has $P$) because this derivation is for the loss function in period $t+1$, rather than for the loss function in period $t$.

Note also that the MAPS normalization that the covariance matrix of the shocks is an identity implies one further difference relative to Dennis (2007). Denoting the covariance matrix of the shocks as $\Omega$ as in Dennis (2007), the definition for the constant term would become (with all else identical):

$$V_{cc} = \frac{1}{1-\beta} tr \left[ \left( \left(\Phi_{r\tilde{z}}\right)' Q \Phi_{r\tilde{z}} + \left(\Phi_{\tilde{x}\tilde{z}}\right)' P \Phi_{\tilde{x}\tilde{z}} \right) \Omega \right]$$

# B Derivation of optimal discretion solution with anticipated disturbances

This appendix presents the derivation of the algorithm used to solve a model under optimal discretion with anticipated disturbances. As noted in Section 4, the derivation proceeds by backward induction. The policy problem is solved in period $H$ before moving back to period $H-1$ and so on. The solution leverages the fact that from period $H$ onwards there are no anticipated disturbances, so the equilibrium is described by the solution derived by Dennis (2007) and presented (in notation consistent with this paper) in Appendix A.

## B.1 Period $H$

In period $H$ the anticipated disturbances are equivalent to unanticipated shocks.[50] This means that the following laws of motion, analogous to those derived in Dennis (2007) are applicable (see Appendix A):

$$\widetilde{x}_H = B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-1} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_H \tag{B.1}$$

$$r_H = B_{r\tilde{x}} \widetilde{x}_{H-1} + \Phi_{r\tilde{z}} \widetilde{z}_H \tag{B.2}$$

where:

$$B_{\tilde{x}\tilde{x}} = -\Theta^{-1} \left( \widetilde{H}_{\tilde{x}}^B + \widetilde{H}_r^C B_{r\tilde{x}} \right) \tag{B.3}$$

$$\Phi_{\tilde{x}\tilde{z}} = \Theta^{-1} \left( \Psi_{\tilde{z}} - \widetilde{H}_r^C \Phi_{r\tilde{z}} \right) \tag{B.4}$$

$$B_{r\tilde{x}} = \Delta_r^{-1} \Delta_{\tilde{x}} \tag{B.5}$$

$$\Phi_{r\tilde{z}} = \Delta_r^{-1} \Delta_{\tilde{z}} \tag{B.6}$$

where:

$$\Theta = \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x}} + \widetilde{H}_r^F B_{r\tilde{x}} \tag{B.7}$$

$$\Delta_r = Q + \zeta \widetilde{H}_r^C \tag{B.8}$$

$$\Delta_{\tilde{x}} = -\zeta \widetilde{H}_{\tilde{x}}^B \tag{B.9}$$

[50] Of course, the fact that they were anticipated in previous periods means that the projection in period $H$ would in general be different from the case in which these shocks are entirely unanticipated: the initial conditions $\widetilde{x}_{H-1}$ will be influenced by the anticipation of $\widetilde{z}_H$ in preceding periods.

$$\Delta_{\tilde{z}} = \zeta \widetilde{\Psi}_{\tilde{z}} \tag{B.10}$$

where:

$$\zeta = \left(\Theta^{-1} \widetilde{H}_r^C\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Theta^{-1} \tag{B.11}$$

with:

$$V_{\tilde{x}\tilde{x}} = \left(B_{\tilde{x}\tilde{x}}\right)' W B_{\tilde{x}\tilde{x}} + \left(B_{r\tilde{x}}\right)' Q B_{r\tilde{x}} + \beta \left(B_{\tilde{x}\tilde{x}}\right)' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}} \tag{B.12}$$

## B.2 Period $H-1$

In period $H-1$, we must take account of the fact that the period $H$ shocks, $\widetilde{z}_H$, are anticipated by both the private sector and the policymaker. The (partitioned) set of equations that describe the model are:

$$\widetilde{H}_{\tilde{x}}^F \widetilde{x}_H + \widetilde{H}_{\tilde{x}}^C \widetilde{x}_{H-1} + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{H-2} + \widetilde{H}_r^F r_H + \widetilde{H}_r^C r_{H-1} = \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} \tag{B.13}$$

We can use the period $H$ laws of motion in equations (B.1)-(B.2) to substitute out $\widetilde{x}_H$ and $r_H$ in equation (B.13) and rearrange to get:

$$\widetilde{x}_{H-1} = \Theta^{-1}\left(\widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} - \widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_H - \widetilde{H}_r^F \Phi_{r\tilde{z}} \widetilde{z}_H - \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{H-2} - \widetilde{H}_r^C r_{H-1}\right) \tag{B.14}$$

where $\Theta$ is defined in equation (B.7). Note that the private sector's expectations take into account the optimal behavior of the policymaker in the future *and* the shocks anticipated to arrive in period $H$.

The policymaker's problem is to minimize the loss function subject to the constraint in equation (B.14). This problem can be represented as a Lagrangean:

$$\begin{aligned} &\min_{\widetilde{x}_{H-1}, r_{H-1}} \left(\widetilde{x}_{H-1}\right)' W \left(\widetilde{x}_{H-1}\right) + \left(r_{H-1}\right)' Q \left(r_{H-1}\right) + \beta \mathcal{L}_H \\ &-2\lambda_{H-1}' \left(\widetilde{x}_{H-1} - \Theta^{-1}\left(\widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} - \widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_H - \widetilde{H}_r^F \Phi_{r\tilde{z}} \widetilde{z}_H - \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{H-2} - \widetilde{H}_r^C r_{H-1}\right)\right) \end{aligned}$$

The first-order conditions are:

$$r_{H-1}: \quad 2\left(r_{H-1}\right)' Q + \beta \frac{\partial \mathcal{L}_H}{\partial r_{H-1}} - 2\lambda_{H-1}' \Theta^{-1} \widetilde{H}_r^C = 0 \tag{B.15}$$

$$\widetilde{x}_{H-1}: \quad 2\left(\widetilde{x}_{H-1}\right)' W + \beta \frac{\partial \mathcal{L}_H}{\partial \widetilde{x}_{H-1}} - 2\lambda_{H-1}' = 0 \tag{B.16}$$

$$\lambda_{H-1}: \quad \widetilde{x}_{H-1} - \Theta^{-1}\left(\widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} - \widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_H - \widetilde{H}_r^F \Phi_{r\tilde{z}} \widetilde{z}_H - \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{H-2} - \widetilde{H}_r^C r_{H-1}\right) = 0 \tag{B.17}$$

Appendix C.1 shows that:

$$\mathcal{L}_H = \left(\widetilde{x}_{H-1}\right)' V_{\tilde{x}\tilde{x}} \widetilde{x}_{H-1} + \left(\widetilde{x}_{H-1}\right)' V_{1,\tilde{x}\tilde{z}} \widetilde{z}_H + \left(\widetilde{z}_H\right)' \left(V_{1,\tilde{x}\tilde{z}}\right)' \widetilde{x}_{H-1} + \left(\widetilde{z}_H\right)' V_{11,\tilde{z}\tilde{z}} \widetilde{z}_H$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (B.12) and:

$$V_{1,\tilde{x}\tilde{z}} = \left(B_{\tilde{x}\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Phi_{\tilde{x}\tilde{z}} + \left(B_{r\tilde{x}}\right)' Q \Phi_{r\tilde{z}} \tag{B.18}$$

where $V_{11,\tilde{z}\tilde{z}}$ is undefined because, as a constant, it will not appear in the first order conditions.

It is straightforward to see that $\frac{\partial \mathcal{L}_H}{\partial r_{H-1}} = 0$, which arises as a consequence of the assumption that there are no lags of the policy instruments in the model equations. It is also straightforward to see that:

$$\frac{\partial \mathcal{L}_H}{\partial \widetilde{x}_{H-1}} = 2\left(\widetilde{x}_{H-1}\right)' V_{\tilde{x}\tilde{x}} + 2\left(\widetilde{z}_H\right)' \left(V_{1,\tilde{x}\tilde{z}}\right)'$$

We can substitute this expression into equation (B.16) and rearrange the result for $\lambda'_{H-1}$:

$$\lambda'_{H-1} = (\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x}}) + (\widetilde{z}_H)' \beta (V_{1,\tilde{x}\tilde{z}})'$$

This in turn can be substituted into equation (B.15) to get:

$$\begin{aligned} (r_{H-1})' Q - \left((\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x}}) + (\widetilde{z}_H)' \beta (V_{1,\tilde{x}\tilde{z}})'\right) \Theta^{-1} \widetilde{H}_r^C &= 0 \\ Q r_{H-1} - \left(\Theta^{-1} \widetilde{H}_r^C\right)' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_{H-1} - \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta V_{1,\tilde{x}\tilde{z}} \widetilde{z}_H &= 0 \end{aligned} \quad \text{(B.19)}$$

If the shocks anticipated for period $H$ are zero (or in the absence of anticipated disturbances), then the targeting rule in equation (B.19) is identical to the targeting rule from the Dennis (2007) solution in Appendix A. Otherwise, the targeting rule is modified to include a term in the anticipated disturbances. These disturbances affect outcomes in period $H$ and, therefore, affect losses in period $H$ and onwards. The policymaker is aware of the shocks (they are anticipated) and so optimally takes them into account when setting policy. This demonstrates that the Dennis (2007) targeting rule derived in Appendix A is not valid in projections with anticipated disturbances.

We can then use the constraint as represented by the FOC for the Lagrange multiplier in equation (B.17) to substitute out $\widetilde{x}_{H-1}$ and the expression in equation (B.18) to substitute out $V_{1,\tilde{x}\tilde{z}}$ to get:

$$\begin{aligned} & Q r_{H-1} - \zeta \left(\widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} - \left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\tilde{z}} + \widetilde{H}_r^F \Phi_{r\tilde{z}}\right) \widetilde{z}_H - \widetilde{H}_{\tilde{x}}^B \widetilde{x}_{H-2} - \widetilde{H}_r^C r_{H-1}\right) \\ - & \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta \left((B_{\tilde{x}\tilde{x}})' (W + \beta V_{\tilde{x}\tilde{x}}) \Phi_{\tilde{x}\tilde{z}} + (B_{r\tilde{x}})' Q \Phi_{r\tilde{z}}\right) \widetilde{z}_H = 0 \end{aligned}$$

where $\zeta$ is defined in equation (B.11). This equation can be rearranged to get the following (the logic of which will become clear when it is generalized below):

$$\Delta_r r_{H-1} = \Delta_{\tilde{x}} \widetilde{x}_{H-2} + \Delta_{\tilde{z}} \widetilde{z}_{H-1} + \left(\left(\Delta_{F_{r\tilde{x}}^{ps}} + \Delta_{F_{r\tilde{x}}^{pol}} (W + \beta V_{\tilde{x}\tilde{x}})\right) \Phi_{\tilde{x}\tilde{z}} + \left(\Delta_{F_{rr}^{ps}} + \Delta_{F_{rr}^{pol}} Q\right) \Phi_{r\tilde{z}}\right) \widetilde{z}_H$$

where $\Delta_r$, $\Delta_{\tilde{x}}$ and $\Delta_{\tilde{z}}$ are defined in equations (B.8)-(B.10), and $^{ps}$ and $^{pol}$ denote terms in the anticipated disturbances arising from via private sector agents' expectations and via policymaker optimization respectively with the associated terms defined as:

$$\begin{aligned} \Delta_{F_{r\tilde{x}}^{ps}} &= -\zeta H_{\tilde{x}}^F && \text{(B.20)} \\ \Delta_{F_{r\tilde{x}}^{pol}} &= \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta (B_{\tilde{x}\tilde{x}})' \\ \Delta_{F_{rr}^{ps}} &= -\zeta \widetilde{H}_r^F \\ \Delta_{F_{rr}^{pol}} &= \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta (B_{r\tilde{x}})' && \text{(B.21)} \end{aligned}$$

It immediately follows that:

$$\begin{aligned} r_{H-1} &= B_{r\tilde{x}} \widetilde{x}_{H-2} + \Phi_{r\tilde{z}} \widetilde{z}_{H-1} + \left(\left(F_{1,r\tilde{x}}^{ps} + F_{1,r\tilde{x}}^{pol}\right) \Phi_{\tilde{x}\tilde{z}} + \left(F_{1,rr}^{ps} + F_{1,rr}^{pol}\right) \Phi_{r\tilde{z}}\right) \widetilde{z}_H \\ &= B_{r\tilde{x}} \widetilde{x}_{H-2} + \Phi_{r\tilde{z}} \widetilde{z}_{H-1} + \left(F_{1,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,rr} \Phi_{r\tilde{z}}\right) \widetilde{z}_H && \text{(B.22)} \end{aligned}$$

where $B_{r\tilde{x}}$ and $\Phi_{r\tilde{z}}$ are defined in equations (B.5)-(B.6) and:

$$\begin{aligned} F_{1,r\tilde{x}}^{ps} &= \Delta_r^{-1} \Delta_{F_{r\tilde{x}}^{ps}} && \text{(B.23)} \\ F_{1,r\tilde{x}}^{pol} &= \Delta_r^{-1} \Delta_{F_{r\tilde{x}}^{pol}} (W + \beta V_{\tilde{x}\tilde{x}}) \\ F_{1,rr}^{ps} &= \Delta_r^{-1} \Delta_{F_{rr}^{ps}} \\ F_{1,rr}^{pol} &= \Delta_r^{-1} \Delta_{F_{rr}^{pol}} Q && \text{(B.24)} \end{aligned}$$

Note that the law of motion incorporates the effect of the shocks anticipated for period $H$, but is otherwise unaffected: the formulae for $B_{r\tilde{x}}$ and $\Phi_{r\tilde{z}}$ are identical to the case studied in Appendix A.

It is then straightforward to define the law of motion for $\widetilde{x}_{H-1}$ by substituting the law of motion for $r_{H-1}$ into the the constraint (B.14):

$$\begin{aligned}\widetilde{x}_{H-1} &= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-2}+\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(\left(F^{ps}_{1,\tilde{x}\tilde{x}}+F^{pol}_{1,\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}}+\left(F^{ps}_{1,\tilde{x}r}+F^{pol}_{1,\tilde{x}r}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_H \\ &= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-2}+\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_H \qquad \text{(B.25)}\end{aligned}$$

where $B_{\tilde{x}\tilde{x}}$ and $\Phi_{\tilde{x}\tilde{z}}$ are defined in equations (B.3)-(B.4) and:

$$\begin{aligned}F^{ps}_{1,\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}^F_{\tilde{x}}+\widetilde{H}^C_r F^{ps}_{1,r\tilde{x}}\right) \qquad \text{(B.26)}\\ F^{pol}_{1,\tilde{x}\tilde{x}} &= -\Theta^{-1}\widetilde{H}^C_r F^{pol}_{1,r\tilde{x}} \\ F^{ps}_{1,\tilde{x}r} &= -\Theta^{-1}\left(\widetilde{H}^F_r+\widetilde{H}^C_r F^{ps}_{1,rr}\right) \\ F^{pol}_{1,\tilde{x}r} &= -\Theta^{-1}\widetilde{H}^C_r F^{pol}_{1,rr} \qquad \text{(B.27)}\end{aligned}$$

Again, note that the definitions for $B_{\tilde{x}\tilde{x}}$ and $\Phi_{\tilde{x}\tilde{z}}$ are identical to the case studied in Appendix A.

## B.3 Period $H-2$

As with period $H-1$, we can derive the effective constraint that the policymaker must take into account when optimising in period $H-2$ by substituting the period $H-1$ laws of motion in equations (B.22) and (B.25) in place of expectations for the period $H-1$ outcomes for the instruments and endogenous variables in the (partitioned) model equations and rearrange the result to get:

$$\widetilde{x}_{H-2} = \Theta^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2}-\widetilde{H}^F_{\tilde{x}}\left(\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)\\ -\widetilde{H}^F_r\left(\Phi_{r\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)-\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-3}-\widetilde{H}^C_r r_{H-2}\end{pmatrix} \qquad \text{(B.28)}$$

The policymaker minimises the following:

$$\begin{aligned}&\min_{\widetilde{x}_{H-2},r_{H-2}} \left(\widetilde{x}_{H-2}\right)'W\left(\widetilde{x}_{H-2}\right)+\left(r_{H-2}\right)'Q\left(r_{H-2}\right)+\beta\mathcal{L}_{H-1}\\ &-2\lambda'_{H-2}\left(\widetilde{x}_{H-2}-\Theta^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2}-\widetilde{H}^F_{\tilde{x}}\left(\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)\\ -\widetilde{H}^F_r\left(\Phi_{r\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-3}-\widetilde{H}^C_r r_{H-2}\end{pmatrix}\right)\end{aligned}$$

The first-order conditions are:

$$r_{H-2}: \quad 2\left(r_{H-2}\right)'Q+\beta\frac{\partial\mathcal{L}_{H-1}}{\partial r_{H-2}}-2\lambda'_{H-2}\Theta^{-1}\widetilde{H}^C_r=0 \qquad \text{(B.29)}$$

$$\widetilde{x}_{H-2}: \quad 2\left(\widetilde{x}_{H-2}\right)'W+\beta\frac{\partial\mathcal{L}_{H-1}}{\partial\widetilde{x}_{H-2}}-2\lambda'_{H-2}=0 \qquad \text{(B.30)}$$

$$\lambda_{H-2}: \quad \widetilde{x}_{H-2}-\Theta^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2}-\widetilde{H}^F_{\tilde{x}}\left(\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)\\ -\widetilde{H}^F_r\left(\Phi_{r\tilde{z}}\widetilde{z}_{H-1}+\left(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_H\right)\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-3}-\widetilde{H}^C_r r_{H-2}\end{pmatrix}=0 \qquad \text{(B.31)}$$

Appendix C.2 shows that:

$$\begin{aligned}\mathcal{L}_{H-1} &= \left(\widetilde{x}_{H-2}\right)'V_{\tilde{x}\tilde{x}}\widetilde{x}_{H-2}+\left(\widetilde{x}_{H-2}\right)'V_{1,\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+\left(\widetilde{z}_{H-1}\right)'\left(V_{1,\tilde{x}\tilde{z}}\right)'\widetilde{x}_{H-2}+\left(\widetilde{x}_{H-2}\right)'V_{2,\tilde{x}\tilde{z}}\widetilde{z}_H\\ &+ \left(\widetilde{z}_H\right)'\left(V_{2,\tilde{x}\tilde{z}}\right)'\widetilde{x}_{H-2}+\sum_{i=1}^{2}\sum_{j=1}^{2}\left(\widetilde{z}_{H-2+i}\right)'\beta V_{ij,\tilde{z}\tilde{z}}\widetilde{z}_{H-2+j}\end{aligned}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (B.12), $V_{1,\tilde{x}\tilde{z}}$ in equation (B.18) and:

$$\begin{aligned} V_{2,\tilde{x}\tilde{z}} &= (B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}})+(B_{r\tilde{x}})'Q(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}})+(B_{\tilde{x}\tilde{x}})'\beta V_{1,\tilde{x}\tilde{z}} \\ &= (B_{\tilde{x}\tilde{x}})'\beta(B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})\Phi_{\tilde{x}\tilde{z}}+(B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}) \\ &\quad +(B_{\tilde{x}\tilde{x}})'\beta(B_{r\tilde{x}})'Q\Phi_{r\tilde{z}}+(B_{r\tilde{x}})'Q(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}) \end{aligned} \tag{B.32}$$

and again the constant terms in the anticipated disturbances, $V_{ij,\tilde{z}\tilde{z}}$, are undefined.

It is straightforward to see that $\frac{\partial \mathcal{L}_{H-1}}{\partial r_{H-2}}=0$ and that:

$$\frac{\partial \mathcal{L}_{H-1}}{\partial \widetilde{x}_{H-2}}=2(\widetilde{x}_{H-2})'V_{\tilde{x}\tilde{x}}+2(\widetilde{z}_{H-1})'(V_{1,\tilde{x}\tilde{z}})'+2(\widetilde{z}_{H})'(V_{2,\tilde{x}\tilde{z}})'$$

We can substitute this expression into equation (B.30) and rearrange the result for $\lambda'_{H-2}$:

$$\lambda'_{H-2}=(\widetilde{x}_{H-2})'(W+\beta V_{\tilde{x}\tilde{x}})+(\widetilde{z}_{H-1})'\beta(V_{1,\tilde{x}\tilde{z}})'+(\widetilde{z}_{H})'\beta(V_{2,\tilde{x}\tilde{z}})'$$

This in turn can be substituted into equation (B.29) to get:

$$\begin{aligned} (r_{H-2})'Q-\left((\widetilde{x}_{H-2})'(W+\beta V_{\tilde{x}\tilde{x}})+(\widetilde{z}_{H-1})'\beta(V_{1,\tilde{x}\tilde{z}})'+(\widetilde{z}_{H})'\beta(V_{2,\tilde{x}\tilde{z}})'\right)\Theta^{-1}\widetilde{H}^C_r &= 0 \\ Qr_{H-2}-\left(\Theta^{-1}\widetilde{H}^C_r\right)'(W+\beta V_{\tilde{x}\tilde{x}})\widetilde{x}_{H-2}-\left(\Theta^{-1}\widetilde{H}^C_r\right)'\beta\left(V_{1,\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+V_{2,\tilde{x}\tilde{z}}\widetilde{z}_{H}\right) &= 0 \end{aligned} \tag{B.33}$$

which extends the period $H-1$ result to show that both one- and two-period ahead anticipated disturbances appear in the targeting rule.

We can then use the constraint as represented by the FOC for the Lagrange multiplier in equation (B.31) to substitute out $\widetilde{x}_{H-2}$ and the expressions in equation (B.18) and (B.32) to substitute out $V_{1,\tilde{x}\tilde{z}}$ and $V_{2,\tilde{x}\tilde{z}}$ to get:

$$\begin{aligned} 0 &= Qr_{H-2}-\zeta\begin{pmatrix} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2} \\ -\widetilde{H}^F_{\tilde{x}}\left(\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-1}+(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}})\widetilde{z}_H\right) \\ -\widetilde{H}^F_{r}\left(\Phi_{r\tilde{z}}\widetilde{z}_{H-1}+(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}})\widetilde{z}_H\right) \\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-3}-\widetilde{H}^C_r r_{H-2} \end{pmatrix} \\ &\quad -\left(\Theta^{-1}\widetilde{H}^C_r\right)'\beta\left(\begin{array}{c} \left((B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})\Phi_{\tilde{x}\tilde{z}}+(B_{r\tilde{x}})'Q\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1} \\ +\begin{pmatrix} (B_{\tilde{x}\tilde{x}})'\beta(B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})\Phi_{\tilde{x}\tilde{z}} \\ +(B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}})(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}) \\ +(B_{\tilde{x}\tilde{x}})'\beta(B_{r\tilde{x}})'Q\Phi_{r\tilde{z}}+(B_{r\tilde{x}})'Q(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}) \end{pmatrix}\widetilde{z}_H \end{array}\right) \end{aligned}$$

This equation can be rearranged to get the following:

$$\begin{aligned} \Delta_r r_{H-2} &= \Delta_{\tilde{x}}\widetilde{x}_{H-3}+\Delta_{\tilde{z}}\widetilde{z}_{H-2} \\ &\quad +\left(\left(\Delta_{F^{ps}_{r\tilde{x}}}+\Delta_{F^{pol}_{r\tilde{x}}}(W+\beta V_{\tilde{x}\tilde{x}})\right)\Phi_{\tilde{x}\tilde{z}}+\left(\Delta_{F^{ps}_{rr}}+\Delta_{F^{pol}_{rr}}Q\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1} \\ &\quad +\left(\begin{array}{c} \begin{pmatrix} \Delta_{F^{ps}_{r\tilde{x}}}F_{1,\tilde{x}\tilde{x}}+\Delta_{F^{ps}_{rr}}F_{1,r\tilde{x}}+\Delta_{F^{pol}_{r\tilde{x}}}(W+\beta V_{\tilde{x}\tilde{x}})F_{1,\tilde{x}\tilde{x}} \\ +\Delta_{F^{pol}_{rr}}QF_{1,r\tilde{x}}+\Delta_{F^{pol}_{r\tilde{x}}}\beta(B_{\tilde{x}\tilde{x}})'(W+\beta V_{\tilde{x}\tilde{x}}) \end{pmatrix}\Phi_{\tilde{x}\tilde{z}} \\ +\begin{pmatrix} \Delta_{F^{ps}_{r\tilde{x}}}F_{1,\tilde{x}r}+\Delta_{F^{ps}_{rr}}F_{1,rr}+\Delta_{F^{pol}_{r\tilde{x}}}(W+\beta V_{\tilde{x}\tilde{x}})F_{1,\tilde{x}r} \\ +\Delta_{F^{pol}_{rr}}QF_{1,rr}+\Delta_{F^{pol}_{r\tilde{x}}}\beta(B_{r\tilde{x}})'Q \end{pmatrix}\Phi_{r\tilde{z}} \end{array}\right)\widetilde{z}_H \end{aligned}$$

where $\Delta_r$, $\Delta_{\tilde{x}}$ and $\Delta_{\tilde{z}}$ are defined in equations (B.8)-(B.10), and $\Delta_{F^{ps}_{r\tilde{x}}}$ , $\Delta_{F^{pol}_{r\tilde{x}}}$, $\Delta_{F^{ps}_{rr}}$ and $\Delta_{F^{pol}_{rr}}$ are defined in equations (B.20)-(B.21). It follows that:

$$r_{H-2}=B_{r\tilde{x}}\widetilde{x}_{H-3}+\Phi_{r\tilde{z}}\widetilde{z}_{H-2}+\left(\left(F^{ps}_{1,r\tilde{x}}+F^{pol}_{1,r\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}}+\left(F^{ps}_{1,rr}+F^{pol}_{1,rr}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1}$$

$$
\begin{aligned}
&+\left(\left(F^{ps}_{2,r\tilde{x}}+F^{pol}_{2,r\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}}+\left(F^{ps}_{2,rr}+F^{pol}_{2,rr}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H}\\
&=B_{r\tilde{x}}\widetilde{x}_{H-3}+\Phi_{r\tilde{z}}\widetilde{z}_{H-2}+\left(F_{1,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1}+\left(F_{2,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{2,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H}\\
&=B_{r\tilde{x}}\widetilde{x}_{H-3}+\sum_{i=0}^{2}\left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i} \qquad \text{(B.34)}
\end{aligned}
$$

where $B_{r\tilde{x}}$ and $\Phi_{r\tilde{z}}$ are the same as above, $F_{0,rr}=\mathbb{I}$ and $F_{0,r\tilde{x}}=0$, $F^{ps}_{1,r\tilde{x}}$, $F^{pol}_{1,r\tilde{x}}$, $F^{ps}_{1,rr}$ and $F^{pol}_{1,rr}$ are defined in equations (B.23)-(B.24), and where:

$$
\begin{aligned}
F^{ps}_{2,r\tilde{x}} &= \Delta_r^{-1}\left(\Delta_{F^{ps}_{r\tilde{x}}}F_{1,\tilde{x}\tilde{x}}+\Delta_{F^{ps}_{rr}}F_{1,r\tilde{x}}\right) \qquad \text{(B.35)}\\
F^{pol}_{2,r\tilde{x}} &= \Delta_r^{-1}\left(\Delta_{F^{pol}_{r\tilde{x}}}\left(\beta\left(B_{\tilde{x}\tilde{x}}\right)'\left(W+\beta V_{\tilde{x}\tilde{x}}\right)+\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{1,\tilde{x}\tilde{x}}\right)+\Delta_{F^{pol}_{rr}}QF_{1,r\tilde{x}}\right)\\
F^{ps}_{2,rr} &= \Delta_r^{-1}\left(\Delta_{F^{ps}_{r\tilde{x}}}F_{1,\tilde{x}r}+\Delta_{F^{ps}_{rr}}F_{1,rr}\right)\\
F^{pol}_{2,rr} &= \Delta_r^{-1}\left(\Delta_{F^{pol}_{r\tilde{x}}}\left(\beta\left(B_{r\tilde{x}}\right)'Q+\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{1,\tilde{x}r}\right)+\Delta_{F^{pol}_{rr}}QF_{1,rr}\right) \qquad \text{(B.36)}
\end{aligned}
$$

Notice that the loading coefficients for two-period-ahead shocks arising via both private sector expectations and policy optimization depend on the one-period-ahead coefficients arising from both sources. The private sector correctly takes into account that two-period-ahead anticipated disturbances affect one-period-ahead policy optimization and the policymaker correctly takes into account that two-period-ahead anticipated disturbances affect one-period-ahead private sector behavior via both their own expectations and their rational understanding of how policy will respond.

It is then straightforward to define the period $H-2$ law of motion for $\widetilde{x}$ using the constraint (B.28):

$$
\begin{aligned}
\widetilde{x}_{H-2} &= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-3}+\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-2}+\left(\left(F^{ps}_{1,\tilde{x}\tilde{x}}+F^{pol}_{1,\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}}+\left(F^{ps}_{1,\tilde{x}r}+F^{pol}_{1,\tilde{x}r}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1}\\
&\quad+\left(\left(F^{ps}_{2,\tilde{x}\tilde{x}}+F^{pol}_{2,\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}}+\left(F^{ps}_{2,\tilde{x}r}+F^{pol}_{2,\tilde{x}r}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H}\\
&= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-3}+\Phi_{\tilde{x}\tilde{z}}\widetilde{z}_{H-2}+\left(F_{1,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{1,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-1}+\left(F_{2,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{2,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H}\\
&= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-3}+\sum_{i=0}^{2}\left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i} \qquad \text{(B.37)}
\end{aligned}
$$

where $B_{\tilde{x}\tilde{x}}$ and $\Phi_{\tilde{x}\tilde{z}}$ are the same as above, $F_{0,\tilde{x}\tilde{x}}=\mathbb{I}$ and $F_{0,\tilde{x}r}=0$, $F^{ps}_{1,\tilde{x}\tilde{x}}$, $F^{pol}_{1,\tilde{x}\tilde{x}}$, $F^{ps}_{1,\tilde{x}r}$ and $F^{pol}_{1,\tilde{x}r}$ are defined in equations (B.26)-(B.27), and where:

$$
\begin{aligned}
F^{ps}_{2,\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}^{F}_{\tilde{x}}F_{1,\tilde{x}\tilde{x}}+\widetilde{H}^{F}_{r}F_{1,r\tilde{x}}+\widetilde{H}^{C}_{r}F^{ps}_{2,r\tilde{x}}\right) \qquad \text{(B.38)}\\
F^{pol}_{2,\tilde{x}\tilde{x}} &= -\Theta^{-1}\widetilde{H}^{C}_{r}F^{pol}_{2,r\tilde{x}}\\
F^{ps}_{2,\tilde{x}r} &= -\Theta^{-1}\left(\widetilde{H}^{F}_{\tilde{x}}F_{1,\tilde{x}r}+\widetilde{H}^{F}_{r}F_{1,rr}+\widetilde{H}^{C}_{r}F^{ps}_{2,rr}\right)\\
F^{pol}_{2,\tilde{x}r} &= -\Theta^{-1}\widetilde{H}^{C}_{r}F^{pol}_{2,rr} \qquad \text{(B.39)}
\end{aligned}
$$

## B.4 Period $H-3$

We can substitute the period $H-2$ laws of motion in equations (B.34) and (B.37) in place of expectations for period $H-2$ outcomes in the (partitioned) model equations and rearrange the result to get:

$$
\widetilde{x}_{H-3} = \Theta^{-1}\left(\begin{array}{c}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-3}-\widetilde{H}^{F}_{\tilde{x}}\left(\sum_{i=0}^{2}\left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right)\\ -\widetilde{H}^{F}_{r}\left(\sum_{i=0}^{2}\left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}}+F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right)-\widetilde{H}^{B}_{\tilde{x}}\widetilde{x}_{H-4}-\widetilde{H}^{C}_{r}r_{H-3}\end{array}\right) \qquad \text{(B.40)}
$$

where $F_{0,\tilde{x}r}=0$, $F_{0,r\tilde{x}}=0$, $F_{0,\tilde{x}\tilde{x}}=\mathbb{I}$ and $F_{0,rr}=\mathbb{I}$.

The policymaker solves:

$$\min_{\widetilde{x}_{H-3},r_{H-3}} \left(\widetilde{x}_{H-3}\right)' W \left(\widetilde{x}_{H-3}\right) + \left(r_{H-3}\right)' Q \left(r_{H-3}\right) + \beta \mathcal{L}_{H-2}$$
$$-2\lambda'_{H-3}\left(\widetilde{x}_{H-3} - \Theta^{-1}\left(\begin{array}{c} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-3} - \widetilde{H}^F_{\tilde{x}}\left(\sum_{i=0}^{2}\left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^F_{r}\left(\sum_{i=0}^{2}\left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-4} - \widetilde{H}^C_{r} r_{H-3} \end{array}\right)\right)$$

The first-order conditions are:

$$r_{H-3}: \qquad 2\left(r_{H-3}\right)' Q + \beta\frac{\partial \mathcal{L}_{H-2}}{\partial r_{H-3}} - 2\lambda'_{H-3}\Theta^{-1}\widetilde{H}^C_r = 0 \tag{B.41}$$

$$\widetilde{x}_{H-3}: \qquad 2\left(\widetilde{x}_{H-3}\right)' W + \beta\frac{\partial \mathcal{L}_{H-2}}{\partial \widetilde{x}_{H-3}} - 2\lambda'_{H-3} = 0 \tag{B.42}$$

$$\lambda_{H-3}: \qquad \widetilde{x}_{H-3} - \Theta^{-1}\left(\begin{array}{c} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-3} - \widetilde{H}^F_{\tilde{x}}\left(\sum_{i=0}^{2}\left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^F_{r}\left(\sum_{i=0}^{2}\left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-4} - \widetilde{H}^C_{r} r_{H-3} \end{array}\right) = 0 \tag{B.43}$$

Appendix C.3 shows that:

$$\begin{aligned}\mathcal{L}_{H-2} &= \left(\widetilde{x}_{H-3}\right)' V_{\tilde{x}\tilde{x}}\widetilde{x}_{H-3} + \sum_{i=1}^{3}\left(\widetilde{z}_{H-2+i}\right)'\left(V_{i,\tilde{x}\tilde{z}}\right)'\widetilde{x}_{H-3} + \sum_{i=1}^{3}\left(\widetilde{z}_{H-3+i}\right)'\left(V_{i,\tilde{x}\tilde{z}}\right)'\widetilde{x}_{H-3} \\ &\quad + \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\widetilde{z}_{H-3+i}\right)'\beta V_{ij,\tilde{z}\tilde{z}}\widetilde{z}_{H-3+j}\end{aligned}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (B.12), $V_{1,\tilde{x}\tilde{z}}$ in equation (B.18), $V_{2,\tilde{x}\tilde{z}}$ in equation (B.32) and:

$$\begin{aligned}V_{3,\tilde{x}\tilde{z}} &= \left(B_{\tilde{x}\tilde{x}}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\left(F_{2,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{2,\tilde{x}r}\Phi_{r\tilde{z}}\right) + \left(B_{r\tilde{x}}\right)' Q\left(F_{2,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{2,rr}\Phi_{r\tilde{z}}\right) + \left(B_{\tilde{x}\tilde{x}}\right)'\beta V_{2,\tilde{x}\tilde{z}} \\ &= \sum_{j=0}^{2}\left(\beta\left(B_{\tilde{x}\tilde{x}}\right)'\right)^{2-j}\left(\begin{array}{c}\left(B_{\tilde{x}\tilde{x}}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\left(F_{j,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{j,\tilde{x}r}\Phi_{r\tilde{z}}\right) \\ + \left(B_{r\tilde{x}}\right)' Q\left(F_{j,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{j,rr}\Phi_{r\tilde{z}}\right)\end{array}\right)\end{aligned} \tag{B.44}$$

where note again that $F_{0,\tilde{x}r} = 0$, $F_{0,r\tilde{x}} = 0$, $F_{0,\tilde{x}\tilde{x}} = \mathbb{I}$ and $F_{0,rr} = \mathbb{I}$.

It is straightforward to see that $\frac{\partial \mathcal{L}_{H-2}}{\partial r_{H-3}} = 0$ and that:

$$\frac{\partial \mathcal{L}_{H-2}}{\partial \widetilde{x}_{H-3}} = 2\left(\widetilde{x}_{H-3}\right)' V_{\tilde{x}\tilde{x}} + 2\sum_{i=1}^{3}\left(\widetilde{z}_{H-3+i}\right)'\left(V_{i,\tilde{x}\tilde{z}}\right)'$$

We can substitute this expression into equation (B.42) and rearrange the result for $\lambda'_{H-3}$:

$$\lambda'_{H-3} = \left(\widetilde{x}_{H-3}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right) + \sum_{i=1}^{3}\left(\widetilde{z}_{H-3+i}\right)'\beta V'_{i,\tilde{x}\tilde{z}}$$

This in turn can be substituted into equation (B.41) to get:

$$\begin{aligned}\left(r_{H-3}\right)' Q - \left(\left(\widetilde{x}_{H-3}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right) + \sum_{i=1}^{3}\left(\widetilde{z}_{H-3+i}\right)'\beta V'_{i,\tilde{x}\tilde{z}}\right)\Theta^{-1}\widetilde{H}^C_r &= 0 \\ Q r_{H-3} - \left(\Theta^{-1}\widetilde{H}^C_r\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\widetilde{x}_{H-3} - \left(\Theta^{-1}\widetilde{H}^C_r\right)'\beta\sum_{i=1}^{3} V_{i,\tilde{x}\tilde{z}}\widetilde{z}_{H-3+i} &= 0\end{aligned} \tag{B.45}$$

We can then use the constraint as represented by the FOC for the Lagrange multiplier in equation (B.43) to substitute out $\widetilde{x}_{H-3}$ and the expressions in equation (B.18), (B.32) and (B.44) to substitute out $V_{1,\tilde{x}\tilde{z}}$, $V_{2,\tilde{x}\tilde{z}}$ and $V_{3,\tilde{x}\tilde{z}}$ to get:

$$\begin{aligned} 0 &= Qr_{H-3} - \zeta \begin{pmatrix} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-3} - \widetilde{H}^F_{\tilde{x}} \left(\sum_{i=0}^{2} \left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^F_{r} \left(\sum_{i=0}^{2} \left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i}\right) \\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-4} - \widetilde{H}^C_{r} r_{H-3} \end{pmatrix} - \left(\Theta^{-1}\widetilde{H}^C_r\right)'\beta \\ &\times \left(\sum_{i=0}^{2}\sum_{j=0}^{i} \left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{i-j} \begin{pmatrix} \left(B\tilde{x}\tilde{x}\right)'\left(W+\beta V_{\tilde{x}\tilde{x}}\right)\left(F_{j,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{j,\tilde{x}r}\Phi_{r\tilde{z}}\right) \\ + \left(Br\tilde{x}\right)' Q\left(F_{j,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{j,rr}\Phi_{r\tilde{z}}\right) \end{pmatrix} \widetilde{z}_{H-2+i}\right) \end{aligned}$$

This equation can be rearranged to get the following:

$$\begin{aligned} \Delta_r r_{H-3} &= \Delta_{\tilde{x}}\widetilde{x}_{H-4} + \Delta_{\tilde{z}}\widetilde{z}_{H-3} \\ &+ \sum_{i=0}^{2}\left(\left(\Delta_{F^{ps}_{r\tilde{x}}}F_{i,\tilde{x}\tilde{x}} + \Delta_{F^{ps}_{rr}}F_{i,r\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}} + \left(\Delta_{F^{ps}_{r\tilde{x}}}F_{i,\tilde{x}r} + \Delta_{F^{ps}_{rr}}F_{i,rr}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2+i} \\ &+ \left(\Delta_{F^{pol}_{r\tilde{x}}}\left(W+\beta V_{\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}} + \Delta_{F^{pol}_{rr}}Q\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-2} \\ &+ \sum_{i=1}^{2}\left\{ \begin{array}{c} \left[\begin{array}{c} \Delta_{F^{pol}_{r\tilde{x}}}\begin{pmatrix} \sum_{j=0}^{i}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{i-j}\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{j,\tilde{x}\tilde{x}} \\ +\sum_{j=0}^{i-1}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{i-j-1}\beta\left(Br\tilde{x}\right)'QF_{j,r\tilde{x}} \end{pmatrix} \\ +\Delta_{F^{pol}_{rr}}QF_{i,r\tilde{x}} \end{array}\right]\Phi_{\tilde{x}\tilde{z}} \\ +\left[\begin{array}{c} \Delta_{F^{pol}_{r\tilde{x}}}\begin{pmatrix} \sum_{j=0}^{i}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{i-j}\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{j,\tilde{x}r} \\ +\sum_{j=0}^{i-1}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{i-j-1}\beta\left(Br\tilde{x}\right)'QF_{j,rr} \end{pmatrix} \\ +\Delta_{F^{pol}_{rr}}QF_{i,rr} \end{array}\right]\Phi_{r\tilde{z}} \end{array}\right\}\widetilde{z}_{H-2+i} \end{aligned}$$

where $\Delta_r$, $\Delta_{\tilde{x}}$ and $\Delta_{\tilde{z}}$ are defined in equations (B.8)-(B.10), and $\Delta_{F^{ps}_{r\tilde{x}}}$ , $\Delta_{F^{pol}_{r\tilde{x}}}$, $\Delta_{F^{ps}_{rr}}$ and $\Delta_{F^{pol}_{rr}}$ are defined in equations (B.20)-(B.21). It follows that:

$$\begin{aligned} r_{H-3} &= B_{r\tilde{x}}\widetilde{x}_{H-4} + \sum_{i=0}^{3}\left(\left(F^{ps}_{i,r\tilde{x}} + F^{pol}_{i,r\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}} + \left(F^{ps}_{i,rr} + F^{pol}_{i,rr}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-3+i} \\ &= B_{r\tilde{x}}\widetilde{x}_{H-4} + \sum_{i=0}^{3}\left(F_{i,r\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,rr}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-3+i} \end{aligned} \tag{B.46}$$

where $B_{r\tilde{x}}$ and $\Phi_{r\tilde{z}}$ are as above, and $F^{ps}_{1,r\tilde{x}}$, $F^{pol}_{1,r\tilde{x}}$, $F^{ps}_{1,rr}$ and $F^{pol}_{1,rr}$ are defined in equations (B.23)-(B.24), $F^{ps}_{2,r\tilde{x}}$, $F^{pol}_{2,r\tilde{x}}$, $F^{ps}_{2,rr}$ and $F^{pol}_{2,rr}$ in equations (B.35)-(B.36) and where:

$$\begin{aligned} F^{ps}_{3,r\tilde{x}} &= \Delta_r^{-1}\left(\Delta_{F^{ps}_{r\tilde{x}}}F_{2,\tilde{x}\tilde{x}} + \Delta_{F^{ps}_{rr}}F_{2,r\tilde{x}}\right) \\ F^{pol}_{3,r\tilde{x}} &= \Delta_r^{-1}\left(\Delta_{F^{pol}_{r\tilde{x}}}\begin{pmatrix} \sum_{j=0}^{2}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{2-j}\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{j,\tilde{x}\tilde{x}} \\ +\sum_{j=0}^{1}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{1-j}\beta\left(Br\tilde{x}\right)'QF_{j,r\tilde{x}} \end{pmatrix} + \Delta_{F^{pol}_{rr}}QF_{2,r\tilde{x}}\right) \\ F^{ps}_{3,rr} &= \Delta_r^{-1}\left(\Delta_{F^{ps}_{r\tilde{x}}}F_{2,\tilde{x}r} + \Delta_{F^{ps}_{rr}}F_{2,rr}\right) \\ F^{pol}_{3,rr} &= \Delta_r^{-1}\left(\Delta_{F^{pol}_{r\tilde{x}}}\begin{pmatrix} \sum_{j=0}^{2}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{2-j}\left(W+\beta V_{\tilde{x}\tilde{x}}\right)F_{j,\tilde{x}r} \\ +\sum_{j=0}^{1}\left(\beta\left(B\tilde{x}\tilde{x}\right)'\right)^{1-j}\beta\left(Br\tilde{x}\right)'QF_{j,rr} \end{pmatrix} + \Delta_{F^{pol}_{rr}}QF_{2,rr}\right) \end{aligned}$$

It is then straightforward to define the law of motion for $\widetilde{x}_{H-3}$ using the constraint (B.40):

$$\widetilde{x}_{H-3} = B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-4} + \sum_{i=0}^{3}\left(\left(F^{ps}_{i,\tilde{x}\tilde{x}} + F^{pol}_{i,\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\tilde{z}} + \left(F^{ps}_{i,\tilde{x}r} + F^{pol}_{i,\tilde{x}r}\right)\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-3+i}$$

$$= B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-4} + \sum_{i=0}^{3} \left(F_{i,\tilde{x}\tilde{x}}\Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r}\Phi_{r\tilde{z}}\right)\widetilde{z}_{H-3+i} \tag{B.47}$$

where $B_{\tilde{x}\tilde{x}}$ and $\Phi_{\tilde{x}\tilde{z}}$ are as above, and $F^{ps}_{1,\tilde{x}\tilde{x}}$, $F^{pol}_{1,\tilde{x}\tilde{x}}$, $F^{ps}_{1,\tilde{x}r}$ and $F^{pol}_{1,\tilde{x}r}$ are defined in equations (B.26)-(B.27), $F^{ps}_{2,\tilde{x}\tilde{x}}$, $F^{pol}_{2,\tilde{x}\tilde{x}}$, $F^{ps}_{2,\tilde{x}r}$ and $F^{pol}_{2,\tilde{x}r}$ in equations (B.38)-(B.39) and where:

$$\begin{aligned}
F^{ps}_{3,\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}^F_{\tilde{x}} F_{2,\tilde{x}\tilde{x}} + \widetilde{H}^F_r F_{2,r\tilde{x}} + \widetilde{H}^C_r F^{ps}_{3,r\tilde{x}}\right)\\
F^{pol}_{3,\tilde{x}\tilde{x}} &= -\Theta^{-1}\widetilde{H}^C_r F^{pol}_{3,r\tilde{x}}\\
F^{ps}_{3,\tilde{x}r} &= -\Theta^{-1}\left(\widetilde{H}^F_{\tilde{x}} F_{2,\tilde{x}r} + \widetilde{H}^F_r F_{2,rr} + \widetilde{H}^C_r F^{ps}_{3,rr}\right)\\
F^{pol}_{3,\tilde{x}r} &= -\Theta^{-1}\widetilde{H}^C_r F^{pol}_{3,rr}
\end{aligned}$$

# C Future loss function expressions with anticipated disturbances

This appendix derives expressions for the one-period-ahead loss functions allowing for anticipated disturbances.

## C.1 One period ahead loss function in period $H-1$

We know that from period $H+1$ onwards the environment is identical to a perfect-foresight solution considered in Appendix A. This means that it is helpful to write the period $H$ loss as:

$$\mathcal{L}_H = (\widetilde{x}_H)' W \widetilde{x}_H + (r_H)' Q r_H + \beta \mathcal{L}_{H+1} \tag{C.1}$$

We can write the loss in period $H+1$ as a function of $\widetilde{x}_H$ using the derivation from Appendix A, but ignoring the constant term which drops out in a perfect-foresight environment:

$$\mathcal{L}_{H+1} = (\widetilde{x}_H)' V_{\tilde{x}\tilde{x}} \widetilde{x}_H \tag{C.2}$$

where:

$$V_{\tilde{x}\tilde{x}} = (B_{\tilde{x}\tilde{x}})' W B_{\tilde{x}\tilde{x}} + (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}} \tag{C.3}$$

which means we can substitute out $\mathcal{L}_{H+1}$ in the above expression for $\mathcal{L}_H$ to get:

$$\mathcal{L}_H = (\widetilde{x}_H)' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_H + (r_H)' Q r_H \tag{C.4}$$

We can substitute out $\widetilde{x}_H$ and $r_H$ using the period $H$ laws of motion from equations (B.1)-(B.2) to get:

$$\begin{aligned}
\mathcal{L}_H &= (B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-1} + \Phi_{\tilde{x}\tilde{z}}\widetilde{z}_H)' (W + \beta V_{\tilde{x}\tilde{x}}) (B_{\tilde{x}\tilde{x}}\widetilde{x}_{H-1} + \Phi_{\tilde{x}\tilde{z}}\widetilde{z}_H)\\
&+ (B_{r\tilde{x}}\widetilde{x}_{H-1} + \Phi_{r\tilde{z}}\widetilde{z}_H)' Q (B_{r\tilde{x}}\widetilde{x}_{H-1} + \Phi_{r\tilde{z}}\widetilde{z}_H)\\
&= (\widetilde{x}_{H-1})' V_{\tilde{x}\tilde{x}} \widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' V_{1,\tilde{x}\tilde{z}} \widetilde{z}_H + (\widetilde{z}_H)' (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-1} + (\widetilde{z}_H)' V_{11,\tilde{z}\tilde{z}} \widetilde{z}_H
\end{aligned} \tag{C.5}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (C.3) and:

$$V_{1,\tilde{x}\tilde{z}} = (B_{\tilde{x}\tilde{x}})' (W + \beta V_{\tilde{x}\tilde{x}}) \Phi_{\tilde{x}\tilde{z}} + (B_{r\tilde{x}})' Q \Phi_{r\tilde{z}} \tag{C.6}$$

It is also straightforward to form an expression for $V_{11,\tilde{z}\tilde{z}}$, but this constant term drops out of the policymaker's first order conditions and so is not needed to characterize the equilibrium laws of motion.

## C.2 One period ahead loss function in period $H-2$

We can write the period $H-1$ loss function as expected in period $H-2$ as:

$$\mathcal{L}_{H-1} = (\widetilde{x}_{H-1})' W \widetilde{x}_{H-1} + (r_{H-1})' Q r_{H-1} + \beta \mathcal{L}_H \tag{C.7}$$

We can use the expression derived in Appendix C.1 to substitute out $\mathcal{L}_H$ to get:

$$\begin{aligned} \mathcal{L}_{H-1} &= (\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_{H-1} + (r_{H-1})' Q r_{H-1} \\ &+ (\widetilde{x}_{H-1})' \beta V_{1,\tilde{x}\tilde{z}} \widetilde{z}_H + (\widetilde{z}_H)' \beta (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-1} + (\widetilde{z}_H)' \beta V_{11,\tilde{z}\tilde{z}} \widetilde{z}_H \end{aligned} \tag{C.8}$$

We can substitute out $r_{H-1}$ and $\widetilde{x}_{H-1}$ using the period $H-1$ laws of motion from equations (B.22) and (B.25) to get:

$$\begin{aligned} \mathcal{L}_{H-1} &= (B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (F_{1,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_H)' (W + \beta V_{\tilde{x}\tilde{x}}) \\ &\times (B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (F_{1,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_H) \\ &+ (B_{r\tilde{x}} \widetilde{x}_{H-2} + \Phi_{r\tilde{z}} \widetilde{z}_{H-1} + (F_{1,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,rr} \Phi_{r\tilde{z}}) \widetilde{z}_H)' Q \\ &\times (B_{r\tilde{x}} \widetilde{x}_{H-2} + \Phi_{r\tilde{z}} \widetilde{z}_{H-1} + (F_{1,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,rr} \Phi_{r\tilde{z}}) \widetilde{z}_H) \\ &+ (B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (F_{1,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_H)' \beta V_{1,\tilde{x}\tilde{z}} \widetilde{z}_H \\ &+ (\widetilde{z}_H)' \beta (V_{1,\tilde{x}\tilde{z}})' (B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + \Phi_{\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (F_{1,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_H) + (\widetilde{z}_H)' \beta V_{11,\tilde{z}\tilde{z}} \widetilde{z}_H \\ &= (\widetilde{x}_{H-2})' V_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{1,\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (\widetilde{z}_{H-1})' (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{2,\tilde{x}\tilde{z}} \widetilde{z}_H \\ &+ (\widetilde{z}_H)' (V_{2,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + (\widetilde{z}_{H-1})' \beta V_{11,\tilde{z}\tilde{z}} \widetilde{z}_{H-1} + (\widetilde{z}_{H-1})' \beta V_{12,\tilde{z}\tilde{z}} \widetilde{z}_H + (\widetilde{z}_H)' \beta (V_{12,\tilde{z}\tilde{z}})' \widetilde{z}_{H-1} \\ &+ (\widetilde{z}_H)' \beta V_{22,\tilde{z}\tilde{z}} \widetilde{z}_H \\ &= (\widetilde{x}_{H-2})' V_{\tilde{x}\tilde{x}} \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{1,\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (\widetilde{z}_{H-1})' (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{2,\tilde{x}\tilde{z}} \widetilde{z}_H \\ &+ (\widetilde{z}_H)' (V_{2,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + \sum_{i=1}^{2} \sum_{j=1}^{2} (\widetilde{z}_{H-2+i})' \beta V_{ij,\tilde{z}\tilde{z}} \widetilde{z}_{H-2+j} \end{aligned} \tag{C.9}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (C.3), $V_{1,\tilde{x}\tilde{z}}$ in equation (C.6) and:

$$V_{2,\tilde{x}\tilde{z}} = (B_{\tilde{x}\tilde{x}})' (W + \beta V_{\tilde{x}\tilde{x}}) (F_{1,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,\tilde{x}r} \Phi_{r\tilde{z}}) + (B_{r\tilde{x}})' Q (F_{1,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{1,rr} \Phi_{r\tilde{z}}) + (B_{\tilde{x}\tilde{x}})' \beta V_{1,\tilde{x}\tilde{z}} \tag{C.10}$$

It is also straightforward to form expressions for $V_{11,\tilde{z}\tilde{z}}$, $V_{12,\tilde{z}\tilde{z}}$ and $V_{22,\tilde{z}\tilde{z}}$, but these constant terms drop out of the policymaker's first order conditions and so are not needed to characterize the equilibrium laws of motion.

## C.3 One period ahead loss function in period $H-3$

We can write the period $H-2$ loss function as expected in period $H-3$ as:

$$\mathcal{L}_{H-2} = (\widetilde{x}_{H-2})' W \widetilde{x}_{H-2} + (r_{H-2})' Q r_{H-2} + \beta \mathcal{L}_{H-1} \tag{C.11}$$

We can use the expression derived in Section C.2 to substitute out $\mathcal{L}_{H-1}$ to get:

$$\begin{aligned} \mathcal{L}_{H-2} = &\; (\widetilde{x}_{H-2})' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_{H-2} + (r_{H-2})' Q r_{H-2} \\ &+ (\widetilde{x}_{H-2})' V_{1,\tilde{x}\tilde{z}} \widetilde{z}_{H-1} + (\widetilde{z}_{H-1})' (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{2,\tilde{x}\tilde{z}} \widetilde{z}_H + (\widetilde{z}_H)' (V_{1,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} \\ &+ \sum_{i=1}^{2} \sum_{j=1}^{2} (\widetilde{z}_{H-2+i})' \beta V_{ij,\tilde{z}\tilde{z}} \widetilde{z}_{H-2+j} \end{aligned}$$

$$= (\widetilde{x}_{H-2})' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_{H-2} + (r_{H-2})' Q r_{H-2} + \sum_{i=1}^{2} (\widetilde{x}_{H-2})' V_{i,\tilde{x}\tilde{z}} \widetilde{z}_{H-2+i}$$

$$+ \sum_{i=1}^{2} (\widetilde{z}_{H-2+i})' (V_{i,\tilde{x}\tilde{z}})' \widetilde{x}_{H-2} + \sum_{i=1}^{2}\sum_{j=1}^{2} (\widetilde{z}_{H-2+i})' \beta V_{ij,\tilde{z}\tilde{z}} \widetilde{z}_{H-2+j} \quad \text{(C.12)}$$

We can substitute out $r_{H-2}$ and $\widetilde{x}_{H-2}$ using the period $H-2$ laws of motion from equations (B.34) and (B.37) to get:

$$\begin{aligned}
\mathcal{L}_{H-2} &= \left( B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right)' (W + \beta V_{\tilde{x}\tilde{x}}) \\
&\times \left( B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right) \\
&+ \left( B_{r\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,rr} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right)' Q \\
&\times \left( B_{r\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,rr} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right) \\
&+ \sum_{i=1}^{2} \left( B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right)' V_{i,\tilde{x}\tilde{z}} \widetilde{z}_{H-2+i} \\
&+ \sum_{i=1}^{2} (\widetilde{z}_{H-2+i})' V'_{i,\tilde{x}\tilde{z}} \left( B_{\tilde{x}\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=0}^{2} (F_{i,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{i,\tilde{x}r} \Phi_{r\tilde{z}}) \widetilde{z}_{H-2+i} \right) \\
&+ \sum_{i=1}^{2}\sum_{j=1}^{2} (\widetilde{z}_{H-2+i})' \beta V_{ij,\tilde{z}\tilde{z}} \widetilde{z}_{H-2+j} \\
&= (\widetilde{x}_{H-3})' V_{\tilde{x}\tilde{x}} \widetilde{x}_{H-3} + \sum_{i=1}^{3} (\widetilde{z}_{H-3+i})' (V_{i,\tilde{x}\tilde{z}})' \widetilde{x}_{H-3} + \sum_{i=1}^{3} (\widetilde{z}_{H-3+i})' (V_{i,\tilde{x}\tilde{z}})' \widetilde{x}_{H-3} \\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3} (\widetilde{z}_{H-3+i})' \beta V_{ij,\tilde{z}\tilde{z}} \widetilde{z}_{H-3+j} \qquad \text{(C.13)}
\end{aligned}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (C.3), $V_{1,\tilde{x}\tilde{z}}$ in equation (C.6), $V_{2,\tilde{x}\tilde{z}}$ in equation (C.10) and:

$$V_{3,\tilde{x}\tilde{z}} = (B_{\tilde{x}\tilde{x}})' (W + \beta V_{\tilde{x}\tilde{x}}) (F_{2,\tilde{x}\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{2,\tilde{x}r} \Phi_{r\tilde{z}}) + (B_{r\tilde{x}})' Q (F_{2,r\tilde{x}} \Phi_{\tilde{x}\tilde{z}} + F_{2,rr} \Phi_{r\tilde{z}}) + (B_{\tilde{x}\tilde{x}})' \beta V_{2,\tilde{x}\tilde{z}} \quad \text{(C.14)}$$

It is also straightforward to form expressions for $\{\{V_{ij,\tilde{z}\tilde{z}}\}_{j=1}^{3}\}_{i=1}^{3}$, but these constant terms drop out of the policymaker's FOC and so are not needed to characterise the equilibrium laws of motion.

# D Derivation of optimal discretion with instrument bounds

## D.1 Period $H$

In period $H$, there are no further shocks anticipated to arrive in future, so the laws of motion for the endogenous variables and instruments are the same as that derived in Appendix A for the case with no anticipated disturbances and, consistent with equations (55)-(57):

$$\widetilde{x}_H = B_{\tilde{x}\tilde{x},H} \widetilde{x}_{H-1} + \Xi_{0,\tilde{x}\tilde{z},H} \widetilde{z}_H + \gamma_{\tilde{x},H} \quad \text{(D.1)}$$

$$r_H = B_{r\tilde{x},H} \widetilde{x}_{H-1} + \Xi_{0,r\tilde{z},H} \widetilde{z}_H + \gamma_{r,H} \quad \text{(D.2)}$$

$$\mu_H = B_{\mu\tilde{x},H} \widetilde{x}_{H-1} + \Xi_{0,\mu\tilde{z},H} \widetilde{z}_H + \gamma_{\mu,H} \quad \text{(D.3)}$$

where:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},H} &= -\Theta_H^{-1}\left(\widetilde{H}_{\tilde{x}}^B + \widetilde{H}_r^C B_{r\tilde{x},H}\right)\\
\Xi_{0,\tilde{x}\tilde{z},H} &= \Theta_H^{-1}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}_r^C \Phi_{r\tilde{z},H}\right)\\
\gamma_{\tilde{x},H} &= -\Theta_H^{-1}\left(\widetilde{H}_{\tilde{x}}^F \gamma_{\tilde{x},H+1} + \widetilde{H}_r^F \gamma_{r,H+1} + \widetilde{H}_r^C \gamma_{r,H}\right)\\
B_{r\tilde{x},H} &= \Gamma_{rr,H}\Delta_{\tilde{x},H}\\
\Xi_{0,r\tilde{z},H} &= \Gamma_{rr,H}\Delta_{\tilde{z},H}\\
\gamma_{r,H} &= \Gamma_{rr,H}\Delta_{c,H} + \Gamma_{r\mu,H}\mathbb{J}_H b\\
B_{\mu\tilde{x},H} &= \Gamma_{\mu r,H}\Delta_{\tilde{x},H}\\
\Xi_{0,\mu\tilde{z},H} &= \Gamma_{\mu r,H}\Delta_{\tilde{z},H}\\
\gamma_{\mu,H} &= \Gamma_{\mu r,H}\Delta_{c,H} + \Gamma_{\mu\mu,H}\mathbb{J}_H b
\end{aligned}$$

where $\mathbb{J}_H$ is an $n_\mu \times n_\mu$ indicator matrix describing which of the constraints in binding in period $H$, $b$ is an $n_\mu \times 1$ vector of constants in the instrument bound inequality constraints (54):

$$\begin{aligned}
\Theta_H &= \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x},H+1} + \widetilde{H}_r^F B_{r\tilde{x},H+1}\\
\Gamma_{rr,H} &= \Delta_{r,H}^{-1} - \Delta_{r,H}^{-1} S'\Gamma_{\mu\mu,H}\mathbb{J}_H S\Delta_{r,H}^{-1}\\
\Gamma_{r\mu,H} &= \Delta_{r,H}^{-1} S'\Gamma_{\mu\mu,H}\\
\Gamma_{\mu r,H} &= -\Gamma_{\mu\mu,H}\mathbb{J}_H S\Delta_{r,H}^{-1}\\
\Gamma_{\mu\mu,H} &= \left(\mathbb{I} - \mathbb{J}_H + \mathbb{J}_H S\Delta_{r,H}^{-1} S'\right)^{-1}\\
\Delta_{\tilde{x},H} &= -\zeta_H \widetilde{H}_{\tilde{x}}^B\\
\Delta_{\tilde{z},H} &= \zeta_H \widetilde{\Psi}_{\tilde{z}}\\
\Delta_{c,H} &= \left(\Theta_H^{-1}\widetilde{H}_r^C\right)' \beta V_{\tilde{x}\gamma,H+1} - \zeta_H\left(\widetilde{H}_{\tilde{x}}^F \gamma_{\tilde{x},H+1} + \widetilde{H}_r^F \gamma_{r,H+1}\right)
\end{aligned}$$

where $\mathbb{I}$ is an $n_\mu \times n_\mu$ identity matrix and $S$ is an $n_\mu \times n_r$ matrix of coefficients on the instruments in the inequality constraints (54) and:

$$\begin{aligned}
\Delta_{r,H} &= Q + \zeta_H \widetilde{H}_r^C\\
\zeta_H &= \left(\Theta_H^{-1}\widetilde{H}_r^C\right)' \left(W + \beta V_{\tilde{x}\tilde{x},H+1}\right)\Theta_H^{-1}
\end{aligned}$$

The assumption that the steady state is characterized by a regime in which none of the constraints is binding and that period $H$ is the maximal horizon up to which constraints can bind in a transition back to the steady state regime implies that the following is true:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},H+1} &= B_{\tilde{x}\tilde{x}}\\
B_{r\tilde{x},H+1} &= B_{r\tilde{x}}\\
\gamma_{\tilde{x},H+1} &= 0\\
\gamma_{r,H+1} &= 0\\
V_{\tilde{x}\tilde{x},H+1} &= V_{\tilde{x}\tilde{x}}\\
V_{\tilde{x}\gamma,H+1} &= 0
\end{aligned}$$

where $B_{\tilde{x}\tilde{x}}$, $B_{r\tilde{x}}$ and $V_{\tilde{x}\tilde{x}}$ are the expressions from the unconstrained solution defined in Section B.1.

## D.2 Period $H-1$

The model of the economy is represented by the same set of equations as in the variant of the problem without bound constraints in equation (B.13). Consistent with the definition of optimal discretion and the presence of anticipated disturbances, we can derive the effective constraint on the policymaker using equations (D.1)-(D.2) to substitute out expectations and then rearranging to get:

$$\widetilde{x}_{H-1}=\Theta_{H-1}^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-1}-\widetilde{H}_{\tilde{x}}^{F}\Xi_{0,\tilde{x}\tilde{z},H}\widetilde{z}_{H}-\widetilde{H}_{r}^{F}\Xi_{0,r\tilde{z},H}\widetilde{z}_{H}\\ -\widetilde{H}_{\tilde{x}}^{B}\widetilde{x}_{H-2}-\widetilde{H}_{r}^{C}r_{H-1}-\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},H}-\widetilde{H}_{r}^{F}\gamma_{r,H}\end{pmatrix} \tag{D.4}$$

where:

$$\Theta_{H-1}=\widetilde{H}_{\tilde{x}}^{C}+\widetilde{H}_{\tilde{x}}^{F}B_{\tilde{x}\tilde{x},H}+\widetilde{H}_{r}^{F}B_{r\tilde{x},H} \tag{D.5}$$

Given this constraint imposed by the model (with optimal behavior on the part of the policymaker in the future embedded via agents' expectations), the instrument bound constraints in equation (54) and a recursive representation of the loss function (38), the policymaker's loss-minimization problem can be represented as a Lagrangian in the following way:

$$\min_{r_{H-1},\widetilde{x}_{H-1}}\ \max_{\lambda_{H-1},\mu_{H-1}\geq 0}\left\{\begin{array}{c}(\widetilde{x}_{H-1})'W(\widetilde{x}_{H-1})+(r_{H-1})'Q(r_{H-1})+\beta\mathcal{L}_{H}\\ -2\mu_{H-1}'(Sr_{H-1}-b)\\ -2\lambda_{H-1}'\left(\widetilde{x}_{H-1}-\Theta_{H-1}^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-1}-\widetilde{H}_{\tilde{x}}^{F}\Xi_{0,\tilde{x}\tilde{z},H}\widetilde{z}_{H}-\widetilde{H}_{r}^{F}\Xi_{0,r\tilde{z},H}\widetilde{z}_{H}\\ -\widetilde{H}_{\tilde{x}}^{B}\widetilde{x}_{H-2}-\widetilde{H}_{r}^{C}r_{H-1}\\ -\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},H}-\widetilde{H}_{r}^{F}\gamma_{r,H}\end{pmatrix}\right)\end{array}\right\}$$

This problem has the following first-order conditions:

$$r_{H-1}:\quad 2(r_{H-1})'Q+\beta\frac{\partial\mathcal{L}_{H}}{\partial r_{H-1}}-2\mu_{H-1}'S-2\lambda_{H-1}'\Theta_{H-1}^{-1}\widetilde{H}_{r}^{C}=0 \tag{D.6}$$

$$\widetilde{x}_{H-1}:\quad 2(\widetilde{x}_{H-1})'W+\beta\frac{\partial\mathcal{L}_{H}}{\partial\widetilde{x}_{H-1}}-2\lambda_{H-1}'=0 \tag{D.7}$$

$$\mu_{H-1}:\quad Sr_{H-1}-b=0 \tag{D.8}$$

$$\lambda_{H-1}:\quad \widetilde{x}_{H-1}-\Theta_{H-1}^{-1}\begin{pmatrix}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-1}-\widetilde{H}_{\tilde{x}}^{F}\Xi_{0,\tilde{x}\tilde{z},H}\widetilde{z}_{H}-\widetilde{H}_{r}^{F}\Xi_{0,r\tilde{z},H}\widetilde{z}_{H}\\ -\widetilde{H}_{\tilde{x}}^{B}\widetilde{x}_{H-2}-\widetilde{H}_{r}^{C}r_{H-1}-\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},H}-\widetilde{H}_{r}^{F}\gamma_{r,H}\end{pmatrix}=0 \tag{D.9}$$

Appendix E.1 shows that:

$$\begin{aligned}\mathcal{L}_{H}&=(\widetilde{x}_{H-1})'V_{\tilde{x}\tilde{x},H}\widetilde{x}_{H-1}+(\widetilde{x}_{H-1})'V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_{H}+(V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_{H})'\widetilde{x}_{H-1}+(\widetilde{x}_{H-1})'V_{\tilde{x}\gamma,H}\\&+(V_{\tilde{x}\gamma_{H}})'\widetilde{x}_{H-1}+V_{cc,H}\end{aligned} \tag{D.10}$$

where:

$$\begin{aligned}V_{\tilde{x}\tilde{x},H}&=(B_{\tilde{x}\tilde{x},H})'(W+\beta V_{\tilde{x}\tilde{x}})B_{\tilde{x}\tilde{x},H}+(B_{r\tilde{x},H})'QB_{r\tilde{x},H}\\V_{1,\tilde{x}\tilde{z},H}&=(B_{\tilde{x}\tilde{x},H})'(W+\beta V_{\tilde{x}\tilde{x}})\Phi_{\tilde{x}\tilde{z},H}+(B_{r\tilde{x},H})'Q\Phi_{r\tilde{z},H}\\V_{\tilde{x}\gamma,H}&=(B_{\tilde{x}\tilde{x},H})'(W+\beta V_{\tilde{x}\tilde{x}})\gamma_{\tilde{x},H}+(B_{r\tilde{x},H})'Q\gamma_{r,H}\end{aligned}$$

and where $V_{cc,H}$ is a composite term comprised of terms in the one-period ahead shocks, $\widetilde{z}_{H}$ and the constants, $\gamma_{\tilde{x},H}$ and $\gamma_{r,H}$.

From equation (D.10), it is clear that $\frac{\partial\mathcal{L}_{H}}{\partial r_{H-1}}=0$ and:

$$\frac{\partial\mathcal{L}_{H}}{\partial\widetilde{x}_{H-1}}=2(\widetilde{x}_{H-1})'V_{\tilde{x}\tilde{x},H}+2(V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_{H})'+2(V_{\tilde{x}\gamma,H})'$$

We can substitute the expression for $\frac{\partial \mathcal{L}_H}{\partial \widetilde{x}_{H-1}}$ into the first order condition for the endogenous variables (D.7) and rearrange to derive the following expression for the Lagrange multiplier, $\lambda'_{H-1}$:

$$\lambda'_{H-1} = (\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) + \beta (V_{1,\tilde{x}\tilde{z},H} \widetilde{z}_H + V_{\tilde{x}\gamma,H})'$$

This in turn can be substituted into the first-order condition for the policy rate (D.6) to give:

$$\begin{aligned} (r_{H-1})' Q - \mu'_{H-1} S - \left( (\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) + \beta (V_{1,\tilde{x}\tilde{z},H} \widetilde{z}_H + V_{\tilde{x}\gamma,H})' \right) \Theta^{-1}_{H-1} \widetilde{H}^C_r &= 0 \\ Q r_{H-1} - S' \mu_{H-1} - \left( \Theta^{-1}_{H-1} \widetilde{H}^C_r \right)' ((W + \beta V_{\tilde{x}\tilde{x},H}) \widetilde{x}_{H-1} + \beta (V_{1,\tilde{x}\tilde{z},H} \widetilde{z}_H + V_{\tilde{x}\gamma,H})) &= 0 \end{aligned} \quad \text{(D.11)}$$

Equation (D.11) takes the same form as the targeting rule in the unconstrained case in equation (B.19), but includes additional terms in the event that any of the constraints are binding in period $H-1$ or $H$.

We can use the constraint (D.9) to substitute out $\widetilde{x}_{H-1}$ in the targeting rule (D.11) and rearrange to get:

$$\begin{aligned} Q r_{H-1} - S' \mu_{H-1} &= \zeta_{H-1} \begin{pmatrix} \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-1} - \left( \widetilde{H}^F_{\tilde{x}} \Xi_{0,\tilde{x}\tilde{z},H} + \widetilde{H}^F_r \Xi_{0,r\tilde{z},H} \right) \widetilde{z}_H \\ -\widetilde{H}^B_{\tilde{x}} \widetilde{x}_{H-2} - \widetilde{H}^C_r r_{H-1} - \widetilde{H}^F_{\tilde{x}} \gamma_{\tilde{x},H} - \widetilde{H}^F_r \gamma_{r,H} \end{pmatrix} \\ &+ \left( \Theta^{-1}_{H-1} \widetilde{H}^C_r \right)' \beta (V_{1,\tilde{x}\tilde{z},H} \widetilde{z}_H + V_{\tilde{x}\gamma,H}) \end{aligned}$$

where:

$$\zeta_{H-1} = \left( \Theta^{-1}_{H-1} \widetilde{H}^C_r \right)' (W + \beta V_{\tilde{x}\tilde{x},H}) \Theta^{-1}_{H-1}$$

We can write this as:

$$\Delta_{r,H-1} r_{H-1} - S' \mu_{H-1} = \Delta_{\tilde{x},H-1} \widetilde{x}_{H-2} + \Delta_{\tilde{z}_0,H-1} \widetilde{z}_{H-1} + \Delta_{\tilde{z}_1,H-1} \widetilde{z}_H + \Delta_{c,H-1} \quad \text{(D.12)}$$

where:

$$\begin{aligned} \Delta_{r,H-1} &= Q + \zeta_{H-1} \widetilde{H}^C_r \\ \Delta_{\tilde{x},H-1} &= -\zeta_{H-1} H^B_{\tilde{x}} \\ \Delta_{\tilde{z}_0,H-1} &= \zeta_{H-1} \widetilde{\Psi}_{\tilde{z}} \\ \Delta_{\tilde{z}_1,H-1} &= \left( \Theta^{-1}_{H-1} \widetilde{H}^C_r \right)' \beta V_{1,\tilde{x}\tilde{z},H} - \zeta_{H-1} \left( \widetilde{H}^F_{\tilde{x}} \Xi_{0,\tilde{x}\tilde{z},H} + \widetilde{H}^F_r \Xi_{0,r\tilde{z},H} \right) \\ \Delta_{c,H-1} &= \left( \Theta^{-1}_{H-1} \widetilde{H}^C_r \right)' \beta V_{\tilde{x}\gamma,H} - \zeta_{H-1} \left( \widetilde{H}^F_{\tilde{x}} \gamma_{\tilde{x},H} + \widetilde{H}^F_r \gamma_{r,H} \right) \end{aligned}$$

We define an indicator, $\mathbb{J}_{H-1}$, as an $n_\mu \times n_\mu$ square matrix with unit entries on the diagonal elements indexing a binding constraint in period $H-1$ and zeros elsewhere. Using this indicator we can define the following system from equation (D.12):

$$\begin{aligned} \begin{bmatrix} \Delta_{r,H-1} & -S' \\ \mathbb{J}_{H-1} S & \mathbb{I} - \mathbb{J}_{H-1} \end{bmatrix} \begin{bmatrix} r_{H-1} \\ \mu_{H-1} \end{bmatrix} &= \begin{bmatrix} \Delta_{\tilde{x},H-1} \\ 0 \end{bmatrix} \widetilde{x}_{H-2} + \begin{bmatrix} \Delta_{\tilde{z}_0,H-1} \\ 0 \end{bmatrix} \widetilde{z}_{H-1} + \begin{bmatrix} \Delta_{\tilde{z}_1,H-1} \\ 0 \end{bmatrix} \widetilde{z}_H \\ &+ \begin{bmatrix} \Delta_{c,H-1} \\ \mathbb{J}_{H-1} b \end{bmatrix} \end{aligned} \quad \text{(D.13)}$$

This system jointly determines the solution for the instruments and the Lagrange multipliers (as long as $\Delta_{r,H-1}$ is invertible):

$$r_{H-1} = B_{r\tilde{x},H-1} \widetilde{x}_{H-2} + \Xi_{0,r\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1} \widetilde{z}_H + \gamma_{r,H-1} \quad \text{(D.14)}$$

$$\mu_{H-1} = B_{\mu\tilde{x},H-1} \widetilde{x}_{H-2} + \Xi_{0,\mu\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,\mu\tilde{z},H-1} \widetilde{z}_H + \gamma_{\mu,H-1} \quad \text{(D.15)}$$

where:

$$\begin{aligned}
B_{r\tilde{x},H-1} &= \Gamma_{rr,H-1}\Delta_{\tilde{x},H-1} \\
\Xi_{0,r\tilde{z},H-1} &= \Gamma_{rr,H-1}\Delta_{\tilde{z}_0,H-1} \\
\Xi_{1,r\tilde{z},H-1} &= \Gamma_{rr,H-1}\Delta_{\tilde{z}_1,H-1} \\
\gamma_{r,H-1} &= \Gamma_{rr,H-1}\Delta_{c,H-1} + \Gamma_{r\mu,H-1}\mathbb{J}_{H-1}b \\
B_{\mu\tilde{x},H-1} &= \Gamma_{\mu r,H-1}\Delta_{\tilde{x},H-1} \\
\Xi_{0,\mu\tilde{z},H-1} &= \Gamma_{\mu r,H-1}\Delta_{\tilde{z}_0,H-1} \\
\Xi_{1,\mu\tilde{z},H-1} &= \Gamma_{\mu r,H-1}\Delta_{\tilde{z}_1,H-1} \\
\gamma_{\mu,H-1} &= \Gamma_{\mu r,H-1}\Delta_{c,H-1} + \Gamma_{\mu\mu,H-1}\mathbb{J}_{H-1}b
\end{aligned}$$

where $\Gamma_{rr,H-1}$, $\Gamma_{r\mu,H-1}$, $\Gamma_{\mu r,H-1}$ and $\Gamma_{\mu\mu,H-1}$ are the upper-left, upper-right, lower-left and lower-right blocks of $\begin{bmatrix} \Delta_{r,H-1} & -S' \\ \mathbb{J}_{H-1}S & \mathbb{I}-\mathbb{J}_{H-1} \end{bmatrix}^{-1}$ respectively, defined as:[51]

$$\begin{aligned}
\Gamma_{rr,H-1} &= \Delta_{r,H-1}^{-1} - \Delta_{r,H-1}^{-1}S'\Gamma_{\mu\mu,H-1}\mathbb{J}_{H-1}S\Delta_{r,H-1}^{-1} \\
\Gamma_{r\mu,H-1} &= \Delta_{r,H-1}^{-1}S'\Gamma_{\mu\mu,H-1} \\
\Gamma_{\mu r,H-1} &= -\Gamma_{\mu\mu,H-1}\mathbb{J}_{H-1}S\Delta_{r,H-1}^{-1} \\
\Gamma_{\mu\mu,H-1} &= \left(\mathbb{I} - \mathbb{J}_{H-1} + \mathbb{J}_{H-1}S\Delta_{r,H-1}^{-1}S'\right)^{-1}
\end{aligned}$$

where $\mathbb{I}$ is n $n_\mu \times n_\mu$ identity matrix.

It is then straightforward to characterise the law of motion for the endogenous variables by substituting the law of motion for the instruments into the constraint in equation (D.4):

$$\widetilde{x}_{H-1} = B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + \gamma_{\tilde{x},H-1} \tag{D.16}$$

where:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},H-1} &= -\Theta_{H-1}^{-1}\left(\widetilde{H}_{\tilde{x}}^{B} + \widetilde{H}_{r}^{C}B_{r\tilde{x},H-1}\right) \\
\Xi_{0,\tilde{x}\tilde{z},H-1} &= \Theta_{H-1}^{-1}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}_{r}^{C}\Xi_{0,r\tilde{z},H-1}\right) \\
\Xi_{1,\tilde{x}\tilde{z},H-1} &= -\Theta_{H-1}^{-1}\left(\widetilde{H}_{\tilde{x}}^{F}\Xi_{0,\tilde{x}\tilde{z},H} + \widetilde{H}_{r}^{F}\Xi_{0,r\tilde{z},H} + \widetilde{H}_{r}^{C}\Xi_{1,r\tilde{z},H-1}\right) \\
\gamma_{\tilde{x},H-1} &= -\Theta_{H-1}^{-1}\left(\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},H} + \widetilde{H}_{r}^{F}\gamma_{r,H} + \widetilde{H}_{r}^{C}\gamma_{r,H-1}\right)
\end{aligned}$$

## D.3 Period $H-2$

In period $H-2$, the policymaker and private sector take the policymaker's optimal decision rule in period $H-1$ as given and can observe the disturbances that will be realized in period $H-1$ and $H$. We can embed that in the period $H-2$ problem by substituting out expectations in the model equations using the laws of motion in equations (D.14) and (D.16) and then rearranging to get:

$$\widetilde{x}_{H-2} = \Theta_{H-2}^{-1}\begin{pmatrix} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2} - \widetilde{H}_{\tilde{x}}^{F}\left(\Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H\right) \\ -\widetilde{H}_{r}^{F}\left(\Xi_{0,r\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1}\widetilde{z}_H\right) \\ -\widetilde{H}_{\tilde{x}}^{B}\widetilde{x}_{H-3} - \widetilde{H}_{r}^{C}r_{H-2} - \widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},H-1} - \widetilde{H}_{r}^{F}\gamma_{r,H-1} \end{pmatrix} \tag{D.17}$$

[51] These are the formulae for the block inverse of a $2 \times 2$ matrix. $\Gamma_{\mu\mu,H-1}$ is the Schur complement of the $\Delta_{r,H-1}$ (upper-left) block of the coefficient matrix.

where:

$$\Theta_{H-2} = \widetilde{H}_{\tilde{x}}^{C} + \widetilde{H}_{\tilde{x}}^{F} B_{\tilde{x}\tilde{x},H-1} + \widetilde{H}_{r}^{F} B_{r\tilde{x},H-1} \tag{D.18}$$

Given this constraint and the bound constraints, the policymaker's loss-minimisation problem can be represented as a Lagrangian in the same way as for period $H-1$:

$$\min_{r_{H-2},\widetilde{x}_{H-2}} \max_{\lambda_{H-2},\mu_{H-2}\geq 0} \left\{ \begin{array}{c} (\widetilde{x}_{H-2})' W (\widetilde{x}_{H-2}) + (r_{H-2})' Q (r_{H-2}) + \beta \mathcal{L}_{H-1} \\ -2\mu'_{H-2} (S r_{H-2} - b) \\ -2\lambda'_{H-2} \left( \widetilde{x}_{H-2} - \Theta_{H-2}^{-1} \left( \begin{array}{c} \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-2} \\ -\widetilde{H}_{\tilde{x}}^{F} (\Xi_{0,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H}) \\ -\widetilde{H}_{r}^{F} (\Xi_{0,r\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1} \widetilde{z}_{H}) \\ -\widetilde{H}_{\tilde{x}}^{B} \widetilde{x}_{H-3} - \widetilde{H}_{r}^{C} r_{H-2} \\ -\widetilde{H}_{\tilde{x}}^{F} \gamma_{\tilde{x},H-1} - \widetilde{H}_{r}^{F} \gamma_{r,H-1} \end{array} \right) \right) \end{array} \right\}$$

This problem has the following first-order conditions:

$$r_{H-2}: \qquad 2 (r_{H-2})' Q + \beta \frac{\partial \mathcal{L}_{H-1}}{\partial r_{H-2}} - 2\mu'_{H-2} S - 2\lambda'_{H-2} \Theta_{H-2}^{-1} \widetilde{H}_{r}^{C} = 0 \tag{D.19}$$

$$\widetilde{x}_{H-2}: \qquad 2 (\widetilde{x}_{H-2})' W + \beta \frac{\partial \mathcal{L}_{H-1}}{\partial \widetilde{x}_{H-2}} - 2\lambda'_{H-2} = 0 \tag{D.20}$$

$$\mu_{H-2}: \qquad S r_{H-2} - b = 0 \tag{D.21}$$

$$\lambda_{H-2}: \qquad \widetilde{x}_{H-2} - \Theta_{H-2}^{-1} \left( \begin{array}{c} \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_{H-2} - \widetilde{H}_{\tilde{x}}^{F} (\Xi_{0,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H}) \\ -\widetilde{H}_{r}^{F} (\Xi_{0,r\tilde{z},H-1} \widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1} \widetilde{z}_{H}) \\ -\widetilde{H}_{\tilde{x}}^{B} \widetilde{x}_{H-3} - \widetilde{H}_{r}^{C} r_{H-2} - \widetilde{H}_{\tilde{x}}^{F} \gamma_{\tilde{x},H-1} - \widetilde{H}_{r}^{F} \gamma_{r,H-1} \end{array} \right) = 0 \tag{D.22}$$

Appendix E.2 shows that:

$$\begin{aligned} \mathcal{L}_{H-1} &= (\widetilde{x}_{H-2})' V_{\tilde{x}\tilde{x},H-1} \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1} + (V_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1})' \widetilde{x}_{H-2} \\ &+ (\widetilde{x}_{H-2})' V_{2,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H} + (V_{2,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H})' \widetilde{x}_{H-2} + (\widetilde{x}_{H-2})' V_{\tilde{x}\gamma,H-1} + (V_{\tilde{x}\gamma,H-1})' \widetilde{x}_{H-2} \\ &+ V_{cc,H-1} \end{aligned} \tag{D.23}$$

where:

$$\begin{aligned} V_{\tilde{x}\tilde{x},H-1} &= (B_{\tilde{x}\tilde{x},H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) B_{\tilde{x}\tilde{x},H-1} + (B_{r\tilde{x},H-1})' Q B_{r\tilde{x},H-1} \\ V_{1,\tilde{x}\tilde{z},H-1} &= (B_{\tilde{x}\tilde{x},H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) \Xi_{0,\tilde{x}\tilde{z},H-1} + (B_{r\tilde{x},H-1})' Q \Xi_{0,r\tilde{z},H-1} \\ V_{2,\tilde{x}\tilde{z},H-1} &= (B_{\tilde{x}\tilde{x},H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) \Xi_{1,\tilde{x}\tilde{z},H-1} + (B_{r\tilde{x},H})' Q \Xi_{1,r\tilde{z},H-1} + (B_{\tilde{x}\tilde{x},H-1})' \beta V_{1,\tilde{x}\tilde{z},H} \\ V_{\tilde{x}\gamma,H-1} &= (B_{\tilde{x}\tilde{x},H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) \gamma_{\tilde{x},H-1} + (B_{r\tilde{x},H-1})' Q \gamma_{r,H-1} + (B_{\tilde{x}\tilde{x},H-1})' \beta V_{\tilde{x}\gamma,H} \end{aligned}$$

and where $V_{cc,H-1}$ is a composite comprised of terms in the one-period and two-period ahead anticipated disturbances, $\widetilde{z}_{H-1}$ and $\widetilde{z}_{H}$, and constants, $\gamma_{\tilde{x},H-1}$, $\gamma_{r,H-1}$, $\gamma_{\tilde{x},H}$ and $\gamma_{r,H}$ (and, therefore, independent of $\widetilde{x}_{H-2}$).

From equation (D.23), it is clear that $\frac{\partial \mathcal{L}_{H-1}}{\partial r_{H-2}} = 0$ and:

$$\frac{\partial \mathcal{L}_{H-1}}{\partial \widetilde{x}_{H-2}} = 2 (\widetilde{x}_{H-2})' V_{\tilde{x}\tilde{x},H-1} + 2 (V_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1})' + 2 (V_{2,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H})' + 2 (V_{\tilde{x}\gamma,H-1})'$$

We can substitute the expression for $\frac{\partial \mathcal{L}_{H-1}}{\partial \widetilde{x}_{H-2}}$ into the first order condition for the endogenous variables (D.20) and rearrange to derive the following expression for the Lagrange multiplier, $\lambda'_{H-2}$:

$$\lambda'_{H-2} = (\widetilde{x}_{H-2})' (W + \beta V_{\tilde{x}\tilde{x},H-1}) + \beta (V_{1,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H-1} + V_{2,\tilde{x}\tilde{z},H-1} \widetilde{z}_{H} + V_{\tilde{x}\gamma,H-1})'$$

This in turn can be substituted into the first-order condition for the policy rate (D.19) to give:

$$
\begin{aligned}
(r_{H-2})' Q - \mu'_{H-2} S - \begin{pmatrix} (\widetilde{x}_{H-2})' (W + \beta V_{\tilde{x}\tilde{x},H-1}) \\ +\beta (V_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + V_{2,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + V_{\tilde{x}\gamma,H-1})' \end{pmatrix} \Theta^{-1}_{H-2}\widetilde{H}^C_r &= 0 \\
Q r_{H-2} - S'\mu_{H-2} - \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' \begin{pmatrix} (W + \beta V_{\tilde{x}\tilde{x},H-1})\,\widetilde{x}_{H-2} \\ +\beta (V_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + V_{2,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + V_{\tilde{x}\gamma,H-1}) \end{pmatrix} &= 0
\end{aligned} \tag{D.24}
$$

which is the targeting rule that now includes terms in period $H-1$ and period $H$ anticipated disturbances, as well terms in the event that any of the constraints are binding in period $H-2$, $H-1$ or $H$.

We can use the constraint (D.22) to substitute out $\widetilde{x}_{H-2}$ in the targeting rule (D.24) and rearrange to get:

$$
\begin{aligned}
Q r_{H-2} - S'\mu_{H-2} &= \zeta_{H-2} \begin{pmatrix} \widetilde{\Psi}_{\tilde{z}}\widetilde{z}_{H-2} - \widetilde{H}^F_{\tilde{x}} \begin{pmatrix} \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} \\ +\Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H \end{pmatrix} - \widetilde{H}^F_r \begin{pmatrix} \Xi_{0,r\tilde{z},H-1}\widetilde{z}_{H-1} \\ +\Xi_{1,r\tilde{z},H-1}\widetilde{z}_H \end{pmatrix} \\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{H-3} - \widetilde{H}^C_r r_{H-2} - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},H-1} - \widetilde{H}^F_r\gamma_{r,H-1} \end{pmatrix} \\
&+ \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' \beta (V_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + V_{2,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + V_{\tilde{x}\gamma,H-1})
\end{aligned}
$$

where:

$$
\zeta_{H-2} = \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' (W + \beta V_{\tilde{x}\tilde{x},H-1})\,\Theta^{-1}_{H-2}
$$

We can write this more compactly as:

$$
\Delta_{r,H-2} r_{H-2} - S'\mu_{H-2} = \Delta_{\tilde{x},H-2}\widetilde{x}_{H-3} + \sum_{s=0}^{2} \Delta_{\tilde{z}_s,H-2}\widetilde{z}_{H-2+s} + \Delta_{c,H-2} \tag{D.25}
$$

where:

$$
\begin{aligned}
\Delta_{r,H-2} &= Q + \zeta_{H-2}\widetilde{H}^C_r \\
\Delta_{\tilde{x},H-2} &= -\zeta_{H-2} H^B_{\tilde{x}} \\
\Delta_{\tilde{z}_0,H-2} &= \zeta_{H-2}\widetilde{\Psi}_{\tilde{z}} \\
\Delta_{\tilde{z}_1,H-2} &= \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' \beta V_{1,\tilde{x}\tilde{z},H-1} - \zeta_{H-2}\left(\widetilde{H}^F_{\tilde{x}}\Xi_{0,\tilde{x}\tilde{z},H-1} + \widetilde{H}^F_r\Xi_{0,r\tilde{z},H-1}\right) \\
\Delta_{\tilde{z}_2,H-2} &= \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' \beta V_{2,\tilde{x}\tilde{z},H-1} - \zeta_{H-2}\left(\widetilde{H}^F_{\tilde{x}}\Xi_{1,\tilde{x}\tilde{z},H-1} + \widetilde{H}^F_r\Xi_{1,r\tilde{z},H-1}\right) \\
\Delta_{c,H-2} &= \left(\Theta^{-1}_{H-2}\widetilde{H}^C_r\right)' \beta V_{\tilde{x}\gamma,H-1} - \zeta_{H-2}\left(\widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},H-1} + \widetilde{H}^F_r\gamma_{r,H-1}\right)
\end{aligned}
$$

We can define the following system for the period $H-2$ instruments and Lagrange multipliers from equation (D.25):

$$
\begin{aligned}
\begin{bmatrix} \Delta_{r,H-2} & -S' \\ \mathbb{J}_{H-2}S & \mathbb{I}-\mathbb{J}_{H-2} \end{bmatrix} \begin{bmatrix} r_{H-2} \\ \mu_{H-2} \end{bmatrix} &= \begin{bmatrix} \Delta_{\tilde{x},H-2} \\ 0 \end{bmatrix} \widetilde{x}_{H-3} + \sum_{s=0}^{2} \begin{bmatrix} \Delta_{\tilde{z}_s,H-2} \\ 0 \end{bmatrix} \widetilde{z}_{H-2+s} \\
&+ \begin{bmatrix} \Delta_{c,H-2} \\ \mathbb{J}_{H-2}b \end{bmatrix}
\end{aligned} \tag{D.26}
$$

This system jointly determines the solution for the instruments and the Lagrange multipliers:

$$
r_{H-2} = B_{r\tilde{x},H-2}\widetilde{x}_{H-3} + \sum_{s=0}^{2} \Xi_{s,r\tilde{z},H-2}\widetilde{z}_{H-2+s} + \gamma_{r,H-2} \tag{D.27}
$$

$$\mu_{H-2} = B_{\mu\tilde{x},H-2}\widetilde{x}_{H-3} + \sum_{s=0}^{2} \Xi_{s,\mu\tilde{z},H-2}\widetilde{z}_{H-2+s} + \gamma_{\mu,H-2} \tag{D.28}$$

where:

$$\begin{aligned}
B_{r\tilde{x},H-2} &= \Gamma_{rr,H-2}\Delta_{\tilde{x},H-2} \\
\Xi_{0,r\tilde{z},H-2} &= \Gamma_{rr,H-2}\Delta_{\tilde{z}_0,H-2} \\
\Xi_{1,r\tilde{z},H-2} &= \Gamma_{rr,H-2}\Delta_{\tilde{z}_1,H-2} \\
\Xi_{2,r\tilde{z},H-2} &= \Gamma_{rr,H-2}\Delta_{\tilde{z}_2,H-2} \\
\gamma_{r,H-2} &= \Gamma_{rr,H-2}\Delta_{c,H-2} + \Gamma_{r\mu,H-2}\mathbb{J}_{H-2}b
\end{aligned}$$

$$\begin{aligned}
B_{\mu\tilde{x},H-2} &= \Gamma_{\mu r,H-2}\Delta_{\tilde{x},H-2} \\
\Xi_{0,\mu\tilde{z},H-2} &= \Gamma_{\mu r,H-2}\Delta_{\tilde{z}_0,H-2} \\
\Xi_{1,\mu\tilde{z},H-2} &= \Gamma_{\mu r,H-2}\Delta_{\tilde{z}_1,H-2} \\
\Xi_{2,\mu\tilde{z},H-2} &= \Gamma_{\mu r,H-2}\Delta_{\tilde{z}_2,H-2} \\
\gamma_{\mu,H-2} &= \Gamma_{\mu r,H-2}\Delta_{c,H-2} + \Gamma_{\mu\mu,H-2}\mathbb{J}_{H-2}b
\end{aligned}$$

where $\Gamma_{rr,H-2}$, $\Gamma_{r\mu,H-2}$, $\Gamma_{\mu r,H-2}$ and $\Gamma_{\mu\mu,H-2}$ are the upper-left, upper-right, lower-left and lower-right blocks of $\left[\begin{array}{cc} \Delta_{r,H-2} & -S' \\ \mathbb{J}_{H-2}S & \mathbb{I}-\mathbb{J}_{H-2} \end{array}\right]^{-1}$ respectively, defined as:

$$\begin{aligned}
\Gamma_{rr,H-2} &= \Delta^{-1}_{r,H-2} - \Delta^{-1}_{r,H-2}S'\Gamma_{\mu\mu,H-2}\mathbb{J}_{H-2}S\Delta^{-1}_{r,H-2} \\
\Gamma_{r\mu,H-2} &= \Delta^{-1}_{r,H-2}S'\Gamma_{\mu\mu,H-2} \\
\Gamma_{\mu r,H-2} &= -\Gamma_{\mu\mu,H-2}\mathbb{J}_{H-2}S\Delta^{-1}_{r,H-2} \\
\Gamma_{\mu\mu,H-2} &= \left(\mathbb{I} - \mathbb{J}_{H-2} + \mathbb{J}_{H-2}S\Delta^{-1}_{r,H-2}S'\right)^{-1}
\end{aligned}$$

where $\mathbb{I}$ is an $n_\mu \times n_\mu$ identity matrix.

It is then straightforward to characterize the law of motion for the endogenous variables by substituting the law of motion for the instruments into the constraint in equation (D.17):

$$\widetilde{x}_{H-2} = B_{\tilde{x}\tilde{x},H-2}\widetilde{x}_{H-3} + \sum_{s=0}^{2} \Xi_{s,\tilde{x}\tilde{z},H-2}\widetilde{z}_{H-2+s} + \gamma_{\tilde{x},H-2} \tag{D.29}$$

where:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},H-2} &= -\Theta^{-1}_{H-2}\left(\widetilde{H}^{B}_{\tilde{x}} + \widetilde{H}^{C}_{r}B_{r\tilde{x},H-2}\right) \\
\Xi_{0,\tilde{x}\tilde{z},H-2} &= \Theta^{-1}_{H-2}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}^{C}_{r}\Xi_{0,r\tilde{z},H-2}\right) \\
\Xi_{1,\tilde{x}\tilde{z},H-2} &= -\Theta^{-1}_{H-2}\left(\widetilde{H}^{F}_{\tilde{x}}\Xi_{0,\tilde{x}\tilde{z},H-1} + \widetilde{H}^{F}_{r}\Xi_{0,r\tilde{z},H-1} + \widetilde{H}^{C}_{r}\Xi_{1,r\tilde{z},H-2}\right) \\
\Xi_{2,\tilde{x}\tilde{z},H-2} &= -\Theta^{-1}_{H-2}\left(\widetilde{H}^{F}_{\tilde{x}}\Xi_{1,\tilde{x}\tilde{z},H-1} + \widetilde{H}^{F}_{r}\Xi_{1,r\tilde{z},H-1} + \widetilde{H}^{C}_{r}\Xi_{2,r\tilde{z},H-2}\right) \\
\gamma_{\tilde{x},H-2} &= -\Theta^{-1}_{H-2}\left(\widetilde{H}^{F}_{\tilde{x}}\gamma_{\tilde{x},H-1} + \widetilde{H}^{F}_{r}\gamma_{r,H-1} + \widetilde{H}^{C}_{r}\gamma_{r,H-2}\right)
\end{aligned}$$

## D.4 A generic period, $t$

The preceding steps reveal the laws of motion for a generic period $t$, valid for $t = 1 \ldots H-1$.

The constraint internalized by the policymaker is:

$$\widetilde{x}_t = \Theta_t^{-1}\left(\begin{array}{c}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^F_{\tilde{x}}\sum_{s=0}^{H-t-1}\Xi_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+1+s} - \widetilde{H}^F_r\sum_{s=0}^{H-t-1}\Xi_{s,r\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r r_t - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},t+1} - \widetilde{H}^F_r\gamma_{r,t+1}\end{array}\right) \tag{D.30}$$

where:

$$\Theta_t = \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}}B_{\tilde{x}\tilde{x},t+1} + \widetilde{H}^F_r B_{r\tilde{x},t+1} \tag{D.31}$$

Given this constraint and the bound constraints, the policymaker's loss-minimisation problem can be represented as a Lagrangian:

$$\min_{r_t,\widetilde{x}_t}\max_{\lambda_t,\mu_t\geq 0}\left\{\begin{array}{c}(\widetilde{x}_t)'W(\widetilde{x}_t) + (r_t)'Q(r_t) + \beta\mathcal{L}_{t+1}\\ -2\mu_t'(Sr_t - b)\\ -2\lambda_t'\left(\widetilde{x}_t - \Theta_t^{-1}\left(\begin{array}{c}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^F_{\tilde{x}}\sum_{s=0}^{H-t-1}\Xi_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^F_r\sum_{s=0}^{H-t-1}\Xi_{s,r\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r r_t - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},t+1} - \widetilde{H}^F_r\gamma_{r,t+1}\end{array}\right)\right)\end{array}\right\}$$

This problem has the following first-order conditions:

$$r_t: \quad 2(r_t)'Q + \beta\frac{\partial\mathcal{L}_{t+1}}{\partial r_t} - 2\mu_t'S - 2\lambda_t'\Theta_t^{-1}\widetilde{H}^C_r = 0 \tag{D.32}$$

$$\widetilde{x}_t: \quad 2(\widetilde{x}_t)'W + \beta\frac{\partial\mathcal{L}_{t+1}}{\partial\widetilde{x}_t} - 2\lambda_t' = 0 \tag{D.33}$$

$$\mu_t: \quad Sr_t - b = 0 \tag{D.34}$$

$$\lambda_t: \quad \widetilde{x}_t - \Theta_t^{-1}\left(\begin{array}{c}\widetilde{\Psi}_{\tilde{z}}\widetilde{z}_t - \widetilde{H}^F_{\tilde{x}}\sum_{s=0}^{H-t-1}\Xi_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^F_r\sum_{s=0}^{H-t-1}\Xi_{s,r\tilde{z},t+1}\widetilde{z}_{t+1+s}\\ -\widetilde{H}^B_{\tilde{x}}\widetilde{x}_{t-1} - \widetilde{H}^C_r r_t - \widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},t+1} - \widetilde{H}^F_r\gamma_{r,t+1}\end{array}\right) = 0 \tag{D.35}$$

Appendix E.3 shows that:

$$\begin{aligned}\mathcal{L}_{t+1} &= (\widetilde{x}_t)'V_{\tilde{x}\tilde{x},t+1}\widetilde{x}_t + (\widetilde{x}_t)'\sum_{s=1}^{H-t}V_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+s} + \sum_{s=1}^{H-t}(V_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+s})'\widetilde{x}_t\\ &+ (\widetilde{x}_t)'V_{\tilde{x}\gamma,t+1} + (V_{\tilde{x}\gamma,t+1})'\widetilde{x}_t + V_{cc,t+1}\end{aligned} \tag{D.36}$$

where:

$$V_{\tilde{x}\tilde{x},t+1} = (B_{\tilde{x}\tilde{x},t+1})'(W + \beta V_{\tilde{x}\tilde{x},t+2})B_{\tilde{x}\tilde{x},t+1} + (B_{r\tilde{x},t+1})'QB_{r\tilde{x},t+1} \tag{D.37}$$

$$\begin{aligned}V_{s,\tilde{x}\tilde{z},t+1} &= (B_{\tilde{x}\tilde{x},t+1})'(W + \beta V_{\tilde{x}\tilde{x},t+2})\Xi_{s-1,\tilde{x}\tilde{z},t+1} + (B_{r\tilde{x},t+1})'Q\Xi_{s-1,r\tilde{z},t+1}\\ &\quad + (B_{\tilde{x}\tilde{x},t+1})'\beta V_{s-1,\tilde{x}\tilde{z},t+2}\end{aligned} \tag{D.38}$$

$$V_{\tilde{x}\gamma,t+1} = (B_{\tilde{x}\tilde{x},t+1})'(W + \beta V_{\tilde{x}\tilde{x},t+2})\gamma_{\tilde{x},t+1} + (B_{r\tilde{x},t+1})'Q\gamma_{r,t+1} + (B_{\tilde{x}\tilde{x},t+1})'\beta V_{\tilde{x}\gamma,t+2} \tag{D.39}$$

with $V_{0,\tilde{x}\tilde{z},t+2} = 0$ (relevant in equation (D.38)), and where $V_{cc,t+1}$ is a composite comprised of terms in anticipated disturbances, $\{\widetilde{z}_{t+s}\}_{s=1}^{H-t}$, and constants, $\{\gamma_{\tilde{x},t+s}\}_{s=1}^{H-t}$ and $\{\gamma_{r,t+s}\}_{s=1}^{H-t}$. (and, therefore, independent of $\widetilde{x}_t$). From equation (D.36), it is clear that $\frac{\partial\mathcal{L}_{t+1}}{\partial r_t} = 0$ and:

$$\frac{\partial\mathcal{L}_{t+1}}{\partial\widetilde{x}_t} = 2(\widetilde{x}_t)'V_{\tilde{x}\tilde{x},t+1} + 2\sum_{s=1}^{H-t}(V_{s,\tilde{x}\tilde{z},t+1}\widetilde{z}_{t+s})' + 2(V_{\tilde{x}\gamma,t+1})'$$

We can substitute the expression for $\frac{\partial\mathcal{L}_{t+1}}{\partial\widetilde{x}_t}$ into the first order condition for the endogenous variables

(D.33) and rearrange to derive the following expression for the Lagrange multiplier, $\lambda_t'$:

$$\lambda_t' = (\widetilde{x}_t)' \left( W + \beta V_{\tilde{x}\tilde{x},t+1} \right) + \beta \left( \sum_{s=1}^{H-t} V_{s,\tilde{x}\tilde{z},t+1} \widetilde{z}_{t+s} + V_{\tilde{x}\gamma,t+1} \right)'$$

This in turn can be substituted into the first-order condition for the policy rate (D.32) to give:

$$\begin{aligned} (r_t)' Q - \mu_t' S - \left( \begin{array}{c} (\widetilde{x}_t)' \left( W + \beta V_{\tilde{x}\tilde{x},t+1} \right) \\ +\beta \left( \sum_{s=1}^{H-t} V_{s,\tilde{x}\tilde{z},t+1} \widetilde{z}_{t+s} + V_{\tilde{x}\gamma,t+1} \right)' \end{array} \right) \Theta_t^{-1} \widetilde{H}_r^C &= 0 \\ Q r_t - S' \mu_t - \left( \Theta_t^{-1} \widetilde{H}_r^C \right)' \left( \begin{array}{c} \left( W + \beta V_{\tilde{x}\tilde{x},t+1} \right) \widetilde{x}_t \\ +\beta \left( \sum_{s=1}^{H-t} V_{s,\tilde{x}\tilde{z},t+1} \widetilde{z}_{t+s} + V_{\tilde{x}\gamma,t+1} \right) \end{array} \right) &= 0 \qquad \text{(D.40)} \end{aligned}$$

which is the generic period $h$ targeting rule.

We can use the first-order condition for the Lagrange multiplier (D.35) to substitute out $\widetilde{x}_t$ in the targeting rule (D.40) and rearrange to get:

$$\begin{aligned} Q r_t - S' \mu_t &= \zeta_t \left( \begin{array}{c} \widetilde{\Psi}_{\tilde{z}} \widetilde{z}_t - \widetilde{H}_{\tilde{x}}^F \sum_{s=0}^{H-t-1} \Xi_{s,\tilde{x}\tilde{z},t+1} \widetilde{z}_{t+1+s} - \widetilde{H}_r^F \sum_{s=0}^{H-t-1} \Xi_{s,r\tilde{z},t+1} \widetilde{z}_{t+1+s} \\ -\widetilde{H}_{\tilde{x}}^B \widetilde{x}_{t-1} - \widetilde{H}_r^C r_t - \widetilde{H}_{\tilde{x}}^F \gamma_{\tilde{x},t+1} - \widetilde{H}_r^F \gamma_{r,t+1} \end{array} \right) \\ &+ \left( \Theta_t^{-1} \widetilde{H}_r^C \right)' \beta \left( \sum_{s=1}^{H-t} V_{s,\tilde{x}\tilde{z},t+1} \widetilde{z}_{t+s} + V_{\tilde{x}\gamma,t+1} \right) \end{aligned}$$

where:

$$\zeta_t = \left( \Theta_t^{-1} \widetilde{H}_r^C \right)' \left( W + \beta V_{\tilde{x}\tilde{x},t+1} \right) \Theta_t^{-1} \qquad \text{(D.41)}$$

Again, this can be written more compactly as:

$$\Delta_{r,t} r_t - S' \mu_t = \Delta_{\tilde{x},t} \widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \Delta_{\tilde{z}_s,t} \widetilde{z}_{t+s} + \Delta_{c,t} \qquad \text{(D.42)}$$

where:

$$\begin{aligned} \Delta_{r,t} &= Q + \zeta_t \widetilde{H}_r^C \qquad \text{(D.43)} \\ \Delta_{\tilde{x},t} &= -\zeta_t H_{\tilde{x}}^B \\ \Delta_{\tilde{z}_0,t} &= \zeta_t \widetilde{\Psi}_{\tilde{z}} \\ \Delta_{\tilde{z}_s,t} &= \left( \Theta_t^{-1} \widetilde{H}_r^C \right)' \beta V_{s,\tilde{x}\tilde{z},t+1} - \zeta_t \left( \widetilde{H}_{\tilde{x}}^F \Xi_{s-1,\tilde{x}\tilde{z},t+1} + \widetilde{H}_r^F \Xi_{s-1,r\tilde{z},t+1} \right) \\ \Delta_{c,t} &= \left( \Theta_t^{-1} \widetilde{H}_r^C \right)' \beta V_{\tilde{x}\gamma,t+1} - \zeta_t \left( \widetilde{H}_{\tilde{x}}^F \gamma_{\tilde{x},t+1} + \widetilde{H}_r^F \gamma_{r,t+1} \right) \qquad \text{(D.44)} \end{aligned}$$

We can define the following system for the period $t$ instruments and Lagrange multipliers from equation (D.42), where $\mathbb{J}_t$ is an $n_\mu \times n_\mu$ diagonal matrix indicating which of the constraints is binding in period $t$:

$$\begin{bmatrix} \Delta_{r,t} & -S' \\ \mathbb{J}_t S & \mathbb{I} - \mathbb{J}_t \end{bmatrix} \begin{bmatrix} r_t \\ \mu_t \end{bmatrix} = \begin{bmatrix} \Delta_{\tilde{x},t} \\ 0 \end{bmatrix} \widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \begin{bmatrix} \Delta_{\tilde{z}_s,t} \\ 0 \end{bmatrix} \widetilde{z}_{t+s} + \begin{bmatrix} \Delta_{c,t} \\ \mathbb{J}_t b \end{bmatrix} \qquad \text{(D.45)}$$

This system jointly determines the solution for the instruments and the Lagrange multipliers:

$$r_t = B_{r\tilde{x},t} \widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \Xi_{s,r\tilde{z},t} \widetilde{z}_{t+s} + \gamma_{r,t} \qquad \text{(D.46)}$$

$$\mu_t = B_{\mu\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \Xi_{s,\mu\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\mu,t} \tag{D.47}$$

where:

$$\begin{aligned}
B_{r\tilde{x},t} &= \Gamma_{rr,t}\Delta_{\tilde{x},t} \quad \text{(D.48)}\\
\Xi_{s,r\tilde{z},t} &= \Gamma_{rr,t}\Delta_{\tilde{z}_s,t}\\
\gamma_{r,t} &= \Gamma_{rr,t}\Delta_{c,t} + \Gamma_{r\mu,t}\mathbb{J}_t b\\
B_{\mu\tilde{x},t} &= \Gamma_{\mu r,t}\Delta_{\tilde{x},t}\\
\Xi_{s,\mu\tilde{z},t} &= \Gamma_{\mu r,t}\Delta_{\tilde{z}_s,t}\\
\gamma_{\mu,t} &= \Gamma_{\mu r,t}\Delta_{c,t} + \Gamma_{\mu\mu,t}\mathbb{J}_t b \quad \text{(D.49)}
\end{aligned}$$

where $\Gamma_{rr,h}$, $\Gamma_{r\mu,h}$, $\Gamma_{\mu r,h}$ and $\Gamma_{\mu\mu,h}$ are the upper-left, upper-right, lower-left and lower-right blocks of $\begin{bmatrix} \Delta_{r,h} & -S' \\ \mathbb{J}_h S & \mathbb{I}-\mathbb{J}_h \end{bmatrix}^{-1}$ respectively, defined as:

$$\begin{aligned}
\Gamma_{rr,t} &= \Delta_{r,t}^{-1} - \Delta_{r,t}^{-1}S'\Gamma_{\mu\mu,t}\mathbb{J}_t S\Delta_{r,t}^{-1} \quad \text{(D.50)}\\
\Gamma_{r\mu,t} &= \Delta_{r,t}^{-1}S'\Gamma_{\mu\mu,t}\\
\Gamma_{\mu r,t} &= -\Gamma_{\mu\mu,t}\mathbb{J}_h S\Delta_{r,t}^{-1}\\
\Gamma_{\mu\mu,t} &= \left(\mathbb{I} - \mathbb{J}_t + \mathbb{J}_t S\Delta_{r,t}^{-1}S'\right)^{-1} \quad \text{(D.51)}
\end{aligned}$$

It is then straightforward to characterise the law of motion for the endogenous variables by substituting the law of motion for the instruments into the constraint in equation (D.30):

$$\widetilde{x}_t = B_{\tilde{x}\tilde{x},t}\widetilde{x}_{t-1} + \sum_{s=0}^{H-t} \Xi_{s,\tilde{x}\tilde{z},t}\widetilde{z}_{t+s} + \gamma_{\tilde{x},t} \tag{D.52}$$

where:

$$\begin{aligned}
B_{\tilde{x}\tilde{x},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^B + \widetilde{H}_r^C B_{r\tilde{x},t}\right) \quad \text{(D.53)}\\
\Xi_{0,\tilde{x}\tilde{z},t} &= \Theta_t^{-1}\left(\widetilde{\Psi}_{\tilde{z}} - \widetilde{H}_r^C\Xi_{0,r\tilde{z},t}\right)\\
\Xi_{s,\tilde{x}\tilde{z},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^F\Xi_{s-1,\tilde{x}\tilde{z},t+1} + \widetilde{H}_r^F\Xi_{s-1,r\tilde{z},t+1} + \widetilde{H}_r^C\Xi_{s,r\tilde{z},t}\right)\\
\gamma_{\tilde{x},t} &= -\Theta_t^{-1}\left(\widetilde{H}_{\tilde{x}}^F\gamma_{\tilde{x},t+1} + \widetilde{H}_r^F\gamma_{r,t+1} + \widetilde{H}_r^C\gamma_{r,t}\right) \quad \text{(D.54)}
\end{aligned}$$

# E Future loss function expressions with instrument bounds

This appendix derives expressions for the one-period-ahead loss functions allowing for anticipated disturbances and instrument bound constraints.

## E.1 One period ahead loss function in period $H-1$

We can write the period $H$ loss as:

$$\mathcal{L}_H = (\widetilde{x}_H)' W\widetilde{x}_H + (r_H)' Qr_H + \beta\mathcal{L}_{H+1} \tag{E.1}$$

Given the assumption that there are no constraints binding in any period beyond $H$ (which means that the loss in period $H+1$ is the same as the case without instrument bound constraints), we can write the loss in period $H+1$ as a function of $\widetilde{x}_H$ using the derivation from Appendix A:

$$\mathcal{L}_{H+1} = (\widetilde{x}_H)' V_{\tilde{x}\tilde{x}} (\widetilde{x}_H) \tag{E.2}$$

where $V_{\tilde{x}\tilde{x}}$ is defined in equation (C.3). We can substitute out $\mathcal{L}_{H+1}$ in the above expression for $\mathcal{L}_H$ to get:

$$\mathcal{L}_H = (\widetilde{x}_H)' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_H + (r_H)' Q r_H \tag{E.3}$$

And then we can substitute out $\widetilde{x}_H$ and $r_H$ using the period $H$ laws of motion from equations (D.1)-(D.2) to get:

$$\begin{aligned}
\mathcal{L}_H &= (B_{\tilde{x}\tilde{x},H}\widetilde{x}_{H-1} + \Xi_{0,\tilde{x}\tilde{z},H}\widetilde{z}_H + \gamma_{\tilde{x},H})' (W + \beta V_{\tilde{x}\tilde{x}}) (B_{\tilde{x}\tilde{x},H}\widetilde{x}_{H-1} + \Xi_{0,\tilde{x}\tilde{z},H}\widetilde{z}_H + \gamma_{\tilde{x},H}) \\
&+ (B_{r\tilde{x},H}\widetilde{x}_{H-1} + \Xi_{0,r\tilde{z},H}\widetilde{z}_H + \gamma_{r,H})' Q (B_{r\tilde{x},H}\widetilde{x}_{H-1} + \Xi_{0,r\tilde{z},H}\widetilde{z}_H + \gamma_{r,H}) \\
&= (\widetilde{x}_{H-1})' V_{\tilde{x}\tilde{x},H}\widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H + (V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H)' \widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' V_{\tilde{x}\gamma,H} + (V_{\tilde{x}\gamma,H})' \widetilde{x}_{H-1} \\
&+ V_{1,\gamma\tilde{z},H}\widetilde{z}_H + (V_{1,\gamma\tilde{z},H}\widetilde{z}_H)' + (\widetilde{z}_H)' V_{11,\tilde{z}\tilde{z},H}\widetilde{z}_H + V_{\gamma\gamma,H} \\
&= (\widetilde{x}_{H-1})' V_{\tilde{x}\tilde{x},H}\widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H + (V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H)' \widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' V_{\tilde{x}\gamma,H} + (V_{\tilde{x}\gamma,H})' \widetilde{x}_{H-1} \\
&+ V_{cc,H}
\end{aligned} \tag{E.4}$$

where:

$$\begin{aligned}
V_{\tilde{x}\tilde{x},H} &= (B_{\tilde{x}\tilde{x},H})' (W + \beta V_{\tilde{x}\tilde{x}}) B_{\tilde{x}\tilde{x},H} + (B_{r\tilde{x},H})' Q B_{r\tilde{x},H} && \text{(E.5)} \\
V_{1,\tilde{x}\tilde{z},H} &= (B_{\tilde{x}\tilde{x},H})' (W + \beta V_{\tilde{x}\tilde{x}}) \Xi_{0,\tilde{x}\tilde{z},H} + (B_{r\tilde{x},H})' Q \Xi_{0,r\tilde{z},H} && \text{(E.6)} \\
V_{\tilde{x}\gamma,H} &= (B_{\tilde{x}\tilde{x},H})' (W + \beta V_{\tilde{x}\tilde{x}}) \gamma_{\tilde{x},H} + (B_{r\tilde{x},H})' Q \gamma_{r,H} && \text{(E.7)}
\end{aligned}$$

and where $V_{cc,H}$ is a composite comprised of terms in the one-period ahead anticipated disturbances, $\widetilde{z}_H$, and constants, $\gamma_{\tilde{x},H}$ and $\gamma_{r,H}$ (and, therefore, independent of $\widetilde{x}_{H-1}$).

## E.2 One period ahead loss function in period $H-2$

The period $H-1$ loss can be written as:

$$\mathcal{L}_{H-1} = (\widetilde{x}_{H-1})' W \widetilde{x}_{H-1} + (r_{H-1})' Q r_{H-1} + \beta \mathcal{L}_H \tag{E.8}$$

We can substitute out $\mathcal{L}_H$ using equation (E.4) to get:

$$\begin{aligned}
\mathcal{L}_{H-1} &= (\widetilde{x}_{H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) \widetilde{x}_{H-1} + (r_{H-1})' Q r_{H-1} \\
&+ (\widetilde{x}_{H-1})' \beta V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H + (\beta V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H)' \widetilde{x}_{H-1} + (\widetilde{x}_{H-1})' \beta V_{\tilde{x}\gamma,H} + (\beta V_{\tilde{x}\gamma,H})' \widetilde{x}_{H-1} \\
&+ \beta V_{cc,H}
\end{aligned} \tag{E.9}$$

We can substitute out $r_{H-1}$ and $\widetilde{x}_{H-1}$ using the period $H-1$ laws of motion from equations (D.14) and (D.16) to get:

$$\begin{aligned}
\mathcal{L}_{H-1} = &\ (B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + \gamma_{\tilde{x},H-1})' (W + \beta V_{\tilde{x}\tilde{x},H}) \\
\times &\ (B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + \gamma_{\tilde{x},H-1}) \\
+ &\ (B_{r\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,r\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1}\widetilde{z}_H + \gamma_{r,H-1})' Q \\
\times &\ (B_{r\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,r\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,r\tilde{z},H-1}\widetilde{z}_H + \gamma_{r,H-1}) \\
+ &\ (B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + \gamma_{\tilde{x},H-1})' \beta V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H \\
+ &\ (\beta V_{1,\tilde{x}\tilde{z},H}\widetilde{z}_H)' (B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_H + \gamma_{\tilde{x},H-1})
\end{aligned}$$

$$
\begin{aligned}
&+ \left(B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H} + \gamma_{\tilde{x},H-1}\right)' \beta V_{\tilde{x}\gamma,H} \\
&+ \left(\beta V_{\tilde{x}\gamma,H}\right)' \left(B_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \Xi_{0,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \Xi_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H} + \gamma_{\tilde{x},H-1}\right) \\
&+ \beta V_{cc,H} \\
&= \left(\widetilde{x}_{H-2}\right)' V_{\tilde{x}\tilde{x},H-1}\widetilde{x}_{H-2} + \left(\widetilde{x}_{H-2}\right)' V_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1} + \left(V_{1,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H-1}\right)' \widetilde{x}_{H-2} \\
&+ \left(\widetilde{x}_{H-2}\right)' V_{2,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H} + \left(V_{2,\tilde{x}\tilde{z},H-1}\widetilde{z}_{H}\right)' \widetilde{x}_{H-2} + \left(\widetilde{x}_{H-2}\right)' V_{\tilde{x}\gamma,H-1} + \left(V_{\tilde{x}\gamma,H-1}\right)' \widetilde{x}_{H-2} \\
&+ V_{cc,H-1}
\end{aligned} \tag{E.10}
$$

where:

$$
\begin{aligned}
V_{\tilde{x}\tilde{x},H-1} &= \left(B_{\tilde{x}\tilde{x},H-1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},H}\right) B_{\tilde{x}\tilde{x},H-1} + \left(B_{r\tilde{x},H-1}\right)' Q B_{r\tilde{x},H-1} && \text{(E.11)} \\
V_{1,\tilde{x}\tilde{z},H-1} &= \left(B_{\tilde{x}\tilde{x},H-1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},H}\right) \Xi_{0,\tilde{x}\tilde{z},H-1} + \left(B_{r\tilde{x},H-1}\right)' Q \Xi_{0,r\tilde{z},H-1} && \text{(E.12)} \\
V_{2,\tilde{x}\tilde{z},H-1} &= \left(B_{\tilde{x}\tilde{x},H-1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},H}\right) \Xi_{1,\tilde{x}\tilde{z},H-1} + \left(B_{r\tilde{x},H}\right)' Q \Xi_{1,r\tilde{z},H-1} \\
&\quad + \left(B_{\tilde{x}\tilde{x},H-1}\right)' \beta V_{1,\tilde{x}\tilde{z},H} && \text{(E.13)} \\
V_{\tilde{x}\gamma,H-1} &= \left(B_{\tilde{x}\tilde{x},H-1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},H}\right) \gamma_{\tilde{x},H-1} + \left(B_{r\tilde{x},H-1}\right)' Q \gamma_{r,H-1} + \left(B_{\tilde{x}\tilde{x},H-1}\right)' \beta V_{\tilde{x}\gamma,H} && \text{(E.14)}
\end{aligned}
$$

and where $V_{cc,H-1}$ is a composite comprised of terms in the one-period and two-period ahead anticipated disturbances, $\widetilde{z}_{H-1}$ and $\widetilde{z}_{H}$, and constants, $\gamma_{\tilde{x},H-1}$, $\gamma_{r,H-1}$, $\gamma_{\tilde{x},H}$ and $\gamma_{r,H}$ (and, therefore, independent of $\widetilde{x}_{H-2}$).

## E.3 One period ahead loss function in period $h$

The generic period $h$ expressions for the one-period ahead loss function, valid for period $h = 1 \ldots H-2$ is as follows. The period $h+1$ loss can be written as:

$$
\mathcal{L}_{h+1} = \left(\widetilde{x}_{h+1}\right)' W \widetilde{x}_{h+1} + \left(r_{h+1}\right)' Q r_{h+1} + \beta \mathcal{L}_{h+2} \tag{E.15}
$$

We substitute out $\mathcal{L}_{h+2}$ using a generic variant of equation (E.10) to get:

$$
\begin{aligned}
\mathcal{L}_{h+1} &= \left(\widetilde{x}_{h+1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},h+2}\right) \widetilde{x}_{h+1} + \left(r_{h+1}\right)' Q r_{h+1} \\
&+ \left(\widetilde{x}_{h+1}\right)' \beta \sum_{s=1}^{H-h-1} V_{s,\tilde{x}\tilde{z},h+2}\widetilde{z}_{h+1+s} + \left(\beta \sum_{s=1}^{H-h-1} V_{s,\tilde{x}\tilde{z},h+2}\widetilde{z}_{h+1+s}\right)' \widetilde{x}_{h+1} \\
&+ \left(\widetilde{x}_{h+1}\right)' \beta V_{\tilde{x}\gamma,h+2} + \left(\beta V_{\tilde{x}\gamma,h+2}\right)' \widetilde{x}_{h+1} \\
&+ \beta V_{cc,h+2}
\end{aligned} \tag{E.16}
$$

We can substitute out $r_{h+1}$ and $\widetilde{x}_{h+1}$ using the period $h+1$ laws of motion to get:

$$
\begin{aligned}
\mathcal{L}_{h+1} = &\left(B_{\tilde{x}\tilde{x},h+1}\widetilde{x}_{h} + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\widetilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},h+2}\right) \\
\times &\left(B_{\tilde{x}\tilde{x},h+1}\widetilde{x}_{h} + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\widetilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right) \\
+ &\left(B_{r\tilde{x},h+1}\widetilde{x}_{h} + \sum_{s=0}^{H-h-1} \Xi_{s,r\tilde{z},h+1}\widetilde{z}_{h+1+s} + \gamma_{r,h+1}\right)' Q \\
\times &\left(B_{r\tilde{x},h+1}\widetilde{x}_{h} + \sum_{s=0}^{H-h-1} \Xi_{s,r\tilde{z},h+1}\widetilde{z}_{h+1+s} + \gamma_{r,h+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(B_{\tilde{x}\tilde{x},h+1}\tilde{x}_h + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right)' \beta \sum_{s=1}^{H-h-1} V_{s,\tilde{x}\tilde{z},h+2}\tilde{z}_{h+1+s} \\
&+ \left(\beta \sum_{s=1}^{H-h-1} V_{s,\tilde{x}\tilde{z},h+2}\tilde{z}_{h+1+s}\right)' \left(B_{\tilde{x}\tilde{x},h+1}\tilde{x}_h + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right) \\
&+ \left(B_{\tilde{x}\tilde{x},h+1}\tilde{x}_h + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right)' \beta V_{\tilde{x}\gamma,h+2} \\
&+ \left(\beta V_{\tilde{x}\gamma,h+2}\right)' \left(B_{\tilde{x}\tilde{x},h+1}\tilde{x}_h + \sum_{s=0}^{H-h-1} \Xi_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+1+s} + \gamma_{\tilde{x},h+1}\right) \\
&+ \beta V_{cc,h+2} \\
&= \left(\tilde{x}_h\right)' V_{\tilde{x}\tilde{x},h+1}\tilde{x}_h + \left(\tilde{x}_h\right)' \sum_{s=1}^{H-h} V_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+s} + \sum_{s=1}^{H-h} \left(V_{s,\tilde{x}\tilde{z},h+1}\tilde{z}_{h+s}\right)' \tilde{x}_h \\
&+ \left(\tilde{x}_h\right)' V_{\tilde{x}\gamma,h+1} + \left(V_{\tilde{x}\gamma,h+1}\right)' \tilde{x}_h + V_{cc,h+1} \qquad \text{(E.17)}
\end{aligned}
$$

where:

$$
\begin{aligned}
V_{\tilde{x}\tilde{x},h+1} &= \left(B_{\tilde{x}\tilde{x},h+1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},h+2}\right) B_{\tilde{x}\tilde{x},h+1} + \left(B_{r\tilde{x},h+1}\right)' Q B_{r\tilde{x},h+1} \qquad \text{(E.18)} \\
V_{s,\tilde{x}\tilde{z},h+1} &= \left(B_{\tilde{x}\tilde{x},h+1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},h+2}\right) \Xi_{s-1,\tilde{x}\tilde{z},h+1} + \left(B_{r\tilde{x},h+1}\right)' Q \Xi_{s-1,r\tilde{z},h+1} \\
&+ \left(B_{\tilde{x}\tilde{x},h+1}\right)' \beta V_{s-1,\tilde{x}\tilde{z},h+2} \qquad \text{(E.19)} \\
V_{\tilde{x}\gamma,h+1} &= \left(B_{\tilde{x}\tilde{x},h+1}\right)' \left(W + \beta V_{\tilde{x}\tilde{x},h+2}\right) \gamma_{\tilde{x},h+1} + \left(B_{r\tilde{x},h+1}\right)' Q \gamma_{r,h+1} + \left(B_{\tilde{x}\tilde{x},h+1}\right)' \beta V_{\tilde{x}\gamma,h+2} \qquad \text{(E.20)}
\end{aligned}
$$

with $V_{0,\tilde{x}\tilde{z},h+2} = 0$, and where $V_{cc,h+1}$ is a composite comprised of terms in anticipated disturbances, $\{\tilde{z}_{h+s}\}_{s=1}^{H-h}$, and constants, $\{\gamma_{\tilde{x},h+s}\}_{s=1}^{H-h}$ and $\{\gamma_{r,h+s}\}_{s=1}^{H-h}$ (and, therefore, independent of $\tilde{x}_h$).

# F Applicability of the Holden and Paetz (2012) approach for optimal discretion

This appendix explores the relationship between the Brendon, Paustian, and Yates (2010) ('BPY') and the Holden and Paetz (2012) ('HP') approach to implementing occasionally binding constraints when policy is set optimally under discretion. It should be re-emphasized that this analysis does not represent a criticism of Holden and Paetz (2012): their methods are designed to handle particular cases and they do not claim that optimal discretionary policy is among those cases.

A simple two-period example is used to make the algebra tractable, together with the following simplifying assumptions:

1. There is a single instrument and a single bound constraint.
2. The steady state is characterized by a regime in which the constraint is not binding.
3. There is a unit coefficient on the instrument in the constraint – i.e. that $S = 1$ in the inequality constraint in equation (54).
4. There are no shocks – i.e. the constraints bind because of the value of the initial condition, $\tilde{x}_0$.

The rest of this appendix proceeds as follows. Section F.1 demonstrates that application of the HP routine produces identical results to the BPY routine if the constraint binds in period 1 only. Section F.2 demonstrates that the HP routine fails to replicate BPY in the case where the constraint is expected to bind in period 2 (but not in period 1), except in the case of purely forward-looking models. It then discusses the source of the failure and, on that basis, shows that the HP routine can be 'corrected' to replicate BPY. Unfortunately, the correction requires knowledge of when the constraint is and is not binding and is, therefore, not useful in practice because it would require iterating over guesses for that in exactly the same way as BPY.

## F.1 Constraint binds in period 1 only

Consider first the case in which the constraint binds in period 1 only. In periods 2 and onwards, it is assumed that the constraint does not bind and so the solution in those periods is time invariant and identical to the unconstrained problem (i.e. Dennis (2007)).

### F.1.1 BPY solution

Given an assumption that the constraint binds in period 1 ($\mathbb{J}_1 = 1$), but not in any future period (and given the particular assumptions described above), the BPY solution for the instrument is:

$$\begin{aligned} r_1 &= B_{r\tilde{x},1}\widetilde{x}_0 + \gamma_{r,1} \\ &= b \end{aligned} \tag{F.1}$$

which follows from:

$$B_{r\tilde{x},1} = \Gamma_{rr,1}\Delta_{\tilde{x},1} \tag{F.2}$$

$$\gamma_{r,1} = \Gamma_{rr,1}\Delta_{c,1} + \Gamma_{r\mu,1}\mathbb{J}_1 b \tag{F.3}$$

and:

$$\begin{aligned} \Gamma_{rr,1} &= \Delta_{r,1}^{-1}\left(\mathbb{I} - \left(\mathbb{I} - \mathbb{J}_1 + \mathbb{J}_1\Delta_{r,1}^{-1}\right)^{-1}\mathbb{J}_1\Delta_{r,1}^{-1}\right) \\ &= 0 \end{aligned} \tag{F.4}$$

$$\begin{aligned} \Gamma_{r\mu,1} &= \Delta_{r,1}^{-1}\left(\mathbb{I} - \mathbb{J}_1 + \mathbb{J}_1\Delta_{r,1}^{-1}\right)^{-1} \\ &= 1 \end{aligned} \tag{F.5}$$

$$\begin{aligned} \Delta_{c,1} &= \left(\Theta_1^{-1}\widetilde{H}_r^C\right)'\beta V_{\tilde{x}\gamma,2} - \zeta_t\left(\widetilde{H}_{\tilde{x}}^F\gamma_{\tilde{x},2} + \widetilde{H}_r^F\gamma_{r,2}\right) \\ &= 0 \end{aligned} \tag{F.6}$$

From this, it follows that:

$$\begin{aligned} \widetilde{x}_1 &= B_{\tilde{x}\tilde{x},1}\widetilde{x}_0 + \gamma_{\tilde{x},1} \\ &= -\Theta_1^{-1}\left(\widetilde{H}_{\tilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^C b\right) \\ &= -\Theta^{-1}\left(\widetilde{H}_{\tilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^C b\right) \end{aligned} \tag{F.7}$$

which follows from application of the formulae for $B_{\tilde{x}\tilde{x},t}$ and $\gamma_{\tilde{x},t}$ with $B_{r\tilde{x},t} = 0$ and $\gamma_{r,t} = b$, and from the observation that the constraint does not bind from period 2 onwards, meaning that the solution is identical to the time-invariant unconstrained Dennis (2007) solution so that $\Theta_1 = \Theta$.

It is also possible to derive an expression for the Lagrange multiplier using the same logic:

$$\begin{aligned} \mu_1 &= B_{\mu\tilde{x},1}\widetilde{x}_0 + \gamma_{\mu,1} \\ &= -\Delta_{\tilde{x},1}\widetilde{x}_0 + \Delta_r b \\ &= -\Delta_{\tilde{x}}\widetilde{x}_0 + \Delta_r b \end{aligned} \tag{F.8}$$

which follows from:

$$
\begin{aligned}
B_{\mu\tilde{x},t} &= \Gamma_{\mu r,1}\Delta_{\tilde{x},1} && \text{(F.9)}\\
\gamma_{\mu,1} &= \Gamma_{\mu r,1}\Delta_{c,1}+\Gamma_{\mu\mu,1}\mathbb{J}_1 b && \text{(F.10)}
\end{aligned}
$$

and:

$$
\begin{aligned}
\Gamma_{\mu r,1} &= -\left(\mathbb{I}-\mathbb{J}_1+\mathbb{J}_1\Delta_{r,1}^{-1}\right)^{-1}\mathbb{J}_1\Delta_{r,1}^{-1}\\
&= -1 && \text{(F.11)}\\
\Gamma_{\mu\mu,1} &= \left(\mathbb{I}-\mathbb{J}_1+\mathbb{J}_1\Delta_{r,1}^{-1}\right)^{-1}\\
&= \Delta_{r,1} && \text{(F.12)}
\end{aligned}
$$

and that $\Delta_{c,1}=0$ as demonstrated above and that $\Delta_{\tilde{x},1}=\Delta_{\tilde{x}}$ and $\Delta_{c,1}=\Delta_c$.

### F.1.2 HP-implied solution

As a starting point for application of the HP routine in this example, consider equation (70) which defines a law of motion for the instruments as part of the BPY solution. Given the assumptions (in particular, that $S=1$ and that there are no constraints binding in any future period), the definitions of the coefficients in that equation allow it to be written as:

$$
r_1=\Delta_r^{-1}\Delta_{\tilde{x}}\widetilde{x}_0+\Delta_r^{-1}\mu_1 \quad \text{(F.13)}
$$

Note that time subscripts have been dropped, consistent with the objective for exploring whether use of the HP algorithm would obviate the requirement to use BPY (and the time variation in the laws of motion that comes with it). Indeed, note that the equation can be rewritten as:

$$
r_1=B_{r\tilde{x}}\widetilde{x}_0+\Phi_{r\mu}\mu_1 \quad \text{(F.14)}
$$

where:

$$
\begin{aligned}
B_{r\tilde{x}} &= \Delta_r^{-1}\Delta_{\tilde{x}} && \text{(F.15)}\\
\Phi_{r\mu} &= \Delta_r^{-1} && \text{(F.16)}
\end{aligned}
$$

Since $r_1=b$ by assumption, the equation can be rearranged to define the Lagrange multiplier, $\mu_1$ as:

$$
\mu_1 = -\Delta_{\tilde{x}}\widetilde{x}_0+\Delta_r b \quad \text{(F.17)}
$$

which is identical to the expression from application of the BPY solution above. We also know that:

$$
\widetilde{x}_1=B_{\tilde{x}\tilde{x}}\widetilde{x}_0+\Phi_{\tilde{x}\mu}\mu_1 \quad \text{(F.18)}
$$

where:

$$
\begin{aligned}
B_{\tilde{x}\tilde{x}} &= -\Theta^{-1}\left(\widetilde{H}_{\tilde{x}}^B+\widetilde{H}_r^C B_{r\tilde{x}}\right) && \text{(F.19)}\\
\Phi_{\tilde{x}\mu} &= -\Theta^{-1}\widetilde{H}_r^C\Phi_{r\mu} && \text{(F.20)}
\end{aligned}
$$

which means that:

$$\begin{aligned}
\widetilde{x}_1 &= B_{\tilde{x}\tilde{x}}\widetilde{x}_0 + \Phi_{\tilde{x}\mu}\mu_1 \\
&= -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r B_{r\tilde{x}}\right)\widetilde{x}_0 - \Theta^{-1}\widetilde{H}^C_r \Phi_{r\mu}\mu_1 \\
&= -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r \Delta_r^{-1}\Delta_{\tilde{x}}\right)\widetilde{x}_0 - \Theta^{-1}\widetilde{H}^C_r \Delta_r^{-1}\mu_1 \\
&= -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r \Delta_r^{-1}\Delta_{\tilde{x}}\right)\widetilde{x}_0 - \Theta^{-1}\widetilde{H}^C_r \Delta_r^{-1}\left(\Delta_r b - \Delta_{\tilde{x}}\widetilde{x}_0\right) \\
&= -\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}}\widetilde{x}_0 + \widetilde{H}^C_r b\right) \qquad \text{(F.21)}
\end{aligned}$$

which is identical to the expression from the BPY solution.

This demonstrates that the HP routine can be used to replicate BPY for a single instrument with a single constraint binding for one period.

## F.2 Constraint binds in period 2 only

This section uses the same example, but assumes that the constraint binds in period 2 (and not in period 1). Once period 2 arrives, the solution is the same as that detailed in Section F.1. This is useful since it means that it will be straightforward to substitute out period 1 expectations for period 2 outcomes. It also means that the source of any difference between the BPY solution and application of the HP routine can be attributed to expectations being incorrectly imputed in the HP routine.

### F.2.1 BPY solution

Given an assumption that the constraint binds in period 2, but not period 1 ($\mathbb{J}_1 = 0$ and $\mathbb{J}_2 = 1$), the BPY solution for the instrument is:

$$r_1 = B_{r\tilde{x},1}\widetilde{x}_0 + \gamma_{r,1} \qquad \text{(F.22)}$$

where:

$$\begin{aligned}
B_{r\tilde{x},1} &= \Gamma_{rr,1}\Delta_{\tilde{x},1} \qquad \text{(F.23)} \\
\gamma_{r,1} &= \Gamma_{rr,1}\Delta_{c,1} + \Gamma_{r\mu,1}\mathbb{J}_1 b \\
&= \Gamma_{rr,1}\Delta_{c,1} \qquad \text{(F.24)}
\end{aligned}$$

and:

$$\begin{aligned}
\Gamma_{rr,1} &= \Delta_{r,1}^{-1}\left(\mathbb{I} - \left(\mathbb{I} - \mathbb{J}_1 + \mathbb{J}_1\Delta_{r,1}^{-1}\right)^{-1}\mathbb{J}_1\Delta_{r,1}^{-1}\right) \\
&= \Delta_{r,1}^{-1} \\
&= \left(Q + \zeta_1\widetilde{H}^C_r\right)^{-1} \qquad \text{(F.25)} \\
\Delta_{\tilde{x},1} &= -\zeta_1\widetilde{H}^B_{\tilde{x}} \qquad \text{(F.26)} \\
\Delta_{c,1} &= \left(\Theta_1^{-1}\widetilde{H}^C_r\right)'\beta V_{\tilde{x}\gamma,2} - \zeta_1\left(\widetilde{H}^F_{\tilde{x}}\gamma_{\tilde{x},2} + \widetilde{H}^F_r\gamma_{r,2}\right) \qquad \text{(F.27)}
\end{aligned}$$

and:

$$\begin{aligned}
\zeta_1 &= \left(\Theta_1^{-1}\widetilde{H}^C_r\right)'\left(W + \beta V_{\tilde{x}\tilde{x},2}\right)\Theta_1^{-1} \qquad \text{(F.28)} \\
\Theta_1 &= \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}}B_{\tilde{x}\tilde{x},2} + \widetilde{H}^F_r B_{r\tilde{x},2} \qquad \text{(F.29)}
\end{aligned}$$

and, given that the constraint is binding in period 2, we can use the results from Section F.1.1 to define $B_{r\tilde{x},2}$, $B_{\tilde{x}\tilde{x},2}$, $\gamma_{r,2}$ and $\gamma_{\tilde{x},2}$ as:

$$B_{r\tilde{x},2} = 0 \tag{F.30}$$

$$B_{\tilde{x}\tilde{x},2} = -\Theta^{-1}\widetilde{H}_{\tilde{x}}^{B} \tag{F.31}$$

$$\gamma_{r,2} = b \tag{F.32}$$

$$\gamma_{\tilde{x},2} = -\Theta^{-1}\widetilde{H}_{r}^{C}b \tag{F.33}$$

and, finally, applying the recursive formulae for the loss function coefficients, along with the fact that the constraint is known to not be binding from period 3 and onwards:

$$\begin{aligned} V_{\tilde{x}\tilde{x},2} &= \left(B_{\tilde{x}\tilde{x},2}\right)'\left(W+\beta V_{\tilde{x}\tilde{x},3}\right)B_{\tilde{x}\tilde{x},2}+\left(B_{r\tilde{x},2}\right)'QB_{r\tilde{x},2} \\ &= \left(\Theta^{-1}\widetilde{H}_{\tilde{x}}^{B}\right)'\left(W+\beta V_{\tilde{x}\tilde{x}}\right)\Theta^{-1}\widetilde{H}_{\tilde{x}}^{B} \end{aligned} \tag{F.34}$$

$$\begin{aligned} V_{\tilde{x}\gamma,2} &= \left(B_{\tilde{x}\tilde{x},2}\right)'\left(W+\beta V_{\tilde{x}\tilde{x},3}\right)\gamma_{\tilde{x},2}+\left(B_{r\tilde{x},2}\right)'Q\gamma_{r,2}+\left(B_{\tilde{x}\tilde{x},2}\right)'V_{\tilde{x}\gamma,3} \\ &= \left(\Theta^{-1}\widetilde{H}_{\tilde{x}}^{B}\right)'\left(W+\beta V_{\tilde{x}\tilde{x}}\right)\Theta^{-1}\widetilde{H}_{r}^{C}b \end{aligned} \tag{F.35}$$

Putting this together (i.e. substituting the relevant expressions through) gives:

$$\begin{aligned} B_{r\tilde{x},1} &= \Gamma_{rr,1}\Delta_{\tilde{x},1} \\ &= -\left(Q+\zeta_1\widetilde{H}_{r}^{C}\right)^{-1}\zeta_1\widetilde{H}_{\tilde{x}}^{B} \end{aligned} \tag{F.36}$$

and:

$$\begin{aligned} \gamma_{r,1} &= \Gamma_{rr,1}\Delta_{c,1} \\ &= \left(Q+\zeta_1\widetilde{H}_{r}^{C}\right)^{-1}\left(\left(\Theta_1^{-1}\widetilde{H}_{r}^{C}\right)'\beta V_{\tilde{x}\gamma,2}-\zeta_1\left(\widetilde{H}_{\tilde{x}}^{F}\gamma_{\tilde{x},2}+\widetilde{H}_{r}^{F}\gamma_{r,2}\right)\right) \\ &= \left(Q+\zeta_1\widetilde{H}_{r}^{C}\right)^{-1}\left(\left(\Theta_1^{-1}\widetilde{H}_{r}^{C}\right)'\beta V_{\tilde{x}\gamma,2}+\zeta_1\left(\widetilde{H}_{\tilde{x}}^{F}\Theta^{-1}\widetilde{H}_{r}^{C}-\widetilde{H}_{r}^{F}\right)b\right) \end{aligned} \tag{F.37}$$

The solution for the non-instrument endogenous variables follows (relatively straightforwardly) from this. It is assumed that if the HP algorithm produces an identical solution for the instrument, then it will also produce an identical solution for the other endogenous variables. Conversely, if application of the HP algorithm does not reproduce the solution for the instrument, then the solution must be incorrect.

### F.2.2 HP-implied solution

To isolate the steps that gives rise to the source of the failure of the HP routine to replicate BPY, the HP-implied solution is derived from first principles.

The policymaker's problem is to minimize the loss function subject to the constraint imposed by the model and subject to the 'shadow shocks', which are known to exist in period 2, given the assumption that the instrument is constrained in period 2. The model equations can be written as:

$$\widetilde{H}_{\tilde{x}}^{F}\widetilde{x}_2+\widetilde{H}_{\tilde{x}}^{C}\widetilde{x}_1+\widetilde{H}_{\tilde{x}}^{B}\widetilde{x}_0+\widetilde{H}_{r}^{F}r_2+\widetilde{H}_{r}^{C}r_1=0 \tag{F.38}$$

Given the assumption that the constraint is binding in period 2 only, the law of motion for $\widetilde{x}_2$ and $r_2$ is the same as in the one-period only binding constraint case from Section F.1.2. These observations can be used to substitute out $\widetilde{x}_2$ and $r_2$ and rearrange to get:

$$\widetilde{x}_1 = -\Theta^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right)\mu_2 + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_0 + \widetilde{H}_r^C r_1\right) \tag{F.39}$$

where:

$$\Theta = \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x}} + \widetilde{H}_r^F B_{r\tilde{x}} \tag{F.40}$$

This can be embedded in the policymaker's loss minimization problem:

$$\begin{aligned} &\min_{\tilde{x}_1, r_1} (\widetilde{x}_1)' W (\widetilde{x}_1) + (r_1)' Q (r_1) + \beta \mathcal{L}_2 \\ &-2\lambda_1' \left(\widetilde{x}_1 + \Theta^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right)\mu_2 + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_0 + \widetilde{H}_r^C r_1\right)\right) \end{aligned} \tag{F.41}$$

The first-order conditions are:

$$r_1: \qquad 2(r_1)' Q + \beta \frac{\partial \mathcal{L}_2}{\partial r_1} - 2\lambda_1' \Theta^{-1} \widetilde{H}_r^C = 0 \tag{F.42}$$

$$\widetilde{x}_1: \qquad 2(\widetilde{x}_1)' W + \beta \frac{\partial \mathcal{L}_2}{\partial \widetilde{x}_1} - 2\lambda_1' = 0 \tag{F.43}$$

$$\lambda_1: \qquad \widetilde{x}_1 + \Theta^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right)\mu_2 + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_0 + \widetilde{H}_r^C r_1\right) = 0 \tag{F.44}$$

The results in Appendix E imply that:

$$\mathcal{L}_2 = (\widetilde{x}_1)' V_{\tilde{x}\tilde{x}} \widetilde{x}_1 + (\widetilde{x}_1)' V_{\tilde{x}\mu} \mu_2 + (\mu_2)' (V_{\tilde{x}\mu})' \widetilde{x}_1 + \mathcal{C} \tag{F.45}$$

where:

$$V_{\tilde{x}\tilde{x}} = (B_{\tilde{x}\tilde{x}})' W B_{\tilde{x}\tilde{x}} + (B_{r\tilde{x}})' Q B_{r\tilde{x}} + \beta (B_{\tilde{x}\tilde{x}})' V_{\tilde{x}\tilde{x}} B_{\tilde{x}\tilde{x}} \tag{F.46}$$

$$V_{\tilde{x}\mu} = (B_{\tilde{x}\tilde{x}})' (W + \beta V_{\tilde{x}\tilde{x}}) \Phi_{\tilde{x}\mu} + (B_{r\tilde{x}})' Q \Phi_{r\mu} \tag{F.47}$$

and $\mathcal{C}$ is a constant that is a function of $\mu_2$ only. It is straightforward to see that $\frac{\partial \mathcal{L}_2}{\partial r_1} = 0$. It is also straightforward to see that:

$$\frac{\partial \mathcal{L}_2}{\partial \widetilde{x}_1} = 2(\widetilde{x}_1)' V_{\tilde{x}\tilde{x}} + 2(\mu_2)' (V_{\tilde{x}\mu})' \tag{F.48}$$

This expression can be substituted into the FOC for the endogenous variables and the result rearranged to give an expression for $\lambda_1'$:

$$\lambda_1' = (\widetilde{x}_1)' (W + \beta V_{\tilde{x}\tilde{x}}) + (\mu_2)' \beta (V_{\tilde{x}\mu})' \tag{F.49}$$

This in turn can be substituted into the FOC for the instrument to get:

$$\begin{aligned} (r_1)' Q - \left((\widetilde{x}_1)' (W + \beta V_{\tilde{x}\tilde{x}}) + (\mu_2)' \beta (V_{\tilde{x}\mu})'\right) \Theta^{-1} \widetilde{H}_r^C &= 0 \\ Q r_1 - \left(\Theta^{-1} \widetilde{H}_r^C\right)' (W + \beta V_{\tilde{x}\tilde{x}}) \widetilde{x}_1 - \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta V_{\tilde{x}\mu} \mu_2 &= 0 \end{aligned} \tag{F.50}$$

Substituting for $\widetilde{x}_1$ using the FOC for the Lagrange multiplier (i.e. the constraint implied by the model combined with the period 2 solution) gives:

$$Q r_1 + \zeta \left(\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right)\mu_2 + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_0 + \widetilde{H}_r^C r_1\right) - \left(\Theta^{-1} \widetilde{H}_r^C\right)' \beta V_{\tilde{x}\mu} \mu_2 = 0 \tag{F.51}$$

where:

$$\zeta = \left(\Theta^{-1} \widetilde{H}_r^C\right)' (W + \beta V_{\tilde{x}\tilde{x}}) \Theta^{-1} \tag{F.52}$$

which can be written as:

$$r_1 = B_{r\tilde{x}}\widetilde{x}_0 + \Xi_{r\mu}\mu_2 \tag{F.53}$$

where:

$$B_{r\tilde{x}} = -\left(Q + \zeta\widetilde{H}_r^C\right)^{-1}\zeta\widetilde{H}_{\tilde{x}}^B \tag{F.54}$$

$$\Xi_{r\mu} = \left(Q + \zeta\widetilde{H}_r^C\right)^{-1}\left(\left(\Theta^{-1}\widetilde{H}_r^C\right)'\beta V_{\tilde{x}\mu} - \zeta\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu}\right)\right) \tag{F.55}$$

It is clear that $B_{r\tilde{x}}$ is not the same as $B_{r\tilde{x},1}$ from the BPY solution above. Does this mean that the HP approach cannot replicate the (correct) BPY solution and is, therefore, not a valid method for imposing instrument bound constraint when policy is set optimally under discretion? That is not immediately obvious. In particular, the algebra in Section F.1.2 demonstrated that:

$$\mu_2 = -\Delta_{\tilde{x}}\widetilde{x}_1 + \Delta_r b \tag{F.56}$$

Since $\widetilde{x}_1$ is a function of $\widetilde{x}_0$, it is no longer obvious that the coefficient on $\widetilde{x}_0$ in a law of motion for $r_1$ with the period 2 shadow shocks substituted out (as is implicitly the form of the BPY solution) is necessarily different to the BPY solution. To proceed, the simultaneity between the shadow shocks in period 2 ($\mu_2$) and outcomes in period 1 ($r_1$ and $\widetilde{x}_1$) must be eliminated.

Start by writing $\widetilde{x}_1$ as:

$$\widetilde{x}_1 = B_{\tilde{x}\tilde{x}}\widetilde{x}_0 + \Xi_{\tilde{x}\mu}\mu_2 \tag{F.57}$$

where:

$$B_{\tilde{x}\tilde{x}} = -\Theta^{-1}\left(\widetilde{H}_{\tilde{x}}^B + \widetilde{H}_r^C B_{r\tilde{x}}\right) \tag{F.58}$$

$$\Xi_{\tilde{x}\mu} = -\Theta^{-1}\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu} + \widetilde{H}_r^C\Xi_{r\mu}\right) \tag{F.59}$$

This can be substituted into the expression for $\mu_2$. Doing so and rearranging gives:

$$\mu_2 = -\left(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}\right)^{-1}\left(\Delta_{\tilde{x}}B_{\tilde{x}\tilde{x}}\widetilde{x}_0 - \Delta_r b\right) \tag{F.60}$$

which exploits the assumption that there is only one instrument constraint (i.e. that $n_\mu = 1$). This in turn can be substituted into the law of motion for the instrument to get:

$$r_1 = B_{r\tilde{x},1}^{\dagger}\widetilde{x}_0 + \gamma_{r,1}^{\dagger} \tag{F.61}$$

where:

$$B_{r\tilde{x},1}^{\dagger} = B_{r\tilde{x}} - \Xi_{r\mu}\left(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}\right)^{-1}\Delta_{\tilde{x}}B_{\tilde{x}\tilde{x}} \tag{F.62}$$

$$\gamma_{r,1}^{\dagger} = \Xi_{r\mu}\left(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}\right)^{-1}\Delta_r b \tag{F.63}$$

Thus, the question of whether or not application of HP can replicate BPY boils down to the question of whether or not $B_{r\tilde{x},1}^{\dagger} = B_{r\tilde{x},1}$ and $\gamma_{r,1}^{\dagger} = \gamma_{r,1}$. First consider $\gamma_{r,1}^{\dagger}$ and suppose (temporarily) that $(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}) = 1$. In that case:

$$\begin{aligned}
\gamma_{r,1}^{\dagger} &= \Xi_{r\mu}\Delta_r b \\
&= \left(Q + \zeta\widetilde{H}_r^C\right)^{-1}\left(\left(\Theta^{-1}\widetilde{H}_r^C\right)'\beta V_{\tilde{x}\mu} - \zeta\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu}\right)\right)\Delta_r b \\
&= \left(Q + \zeta\widetilde{H}_r^C\right)^{-1}\left(\left(\Theta^{-1}\widetilde{H}_r^C\right)'\beta V_{\tilde{x}\mu} + \zeta\left(\widetilde{H}_{\tilde{x}}^F\Theta^{-1}\widetilde{H}_r^C\Delta_r^{-1} + \widetilde{H}_r^F\Delta_r^{-1}\right)\right)\Delta_r b
\end{aligned} \tag{F.64}$$

This is similar to the expression for $\gamma_{r,1}$ (reproduced below), but not identical. In particular, it would be identical if: $\Theta = \Theta_1$, $\zeta = \zeta_1$ and $V_{\tilde{x}\mu}\Delta_r b = V_{\tilde{x}\gamma,2}$. It is straightforward to verify that that is not the case.

$$\gamma_{r,1} = \left(Q + \zeta_1 \widetilde{H}_r^C\right)^{-1} \left(\left(\Theta_1^{-1}\widetilde{H}_r^C\right)' \beta V_{\tilde{x}\gamma,2} + \zeta_1 \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b\right)$$

Of course, there is no reason why $(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}) = 1$ and, in general, it will not. A general expression for $\gamma_{r,1}^{\dagger}$ can be derived by substituting in the expression for $\Xi_{\tilde{x}\mu}$. That expression is clearly different to the correct BPY solution. A similar conclusion can be drawn from comparison of $B_{r\tilde{x},1}^{\dagger}$ with $B_{r\tilde{x},1}$. This demonstrates for this example that application of the HP routine to the problem of optimal discretionary policy with instrument bound constraints will not yield the correct solution.

### F.2.3 A special case

The analysis above suggests that there is a special case in which application of the HP algorithm correctly replicates the BPY solution. That is the case of a purely forward-looking model or, more precisely, a model in which the optimal discretionary targeting rule is static (i.e. one in which there is no effect of choices today on losses in the future). In this special case, $B_{r\tilde{x},1} = B_{r\tilde{x},1}^{\dagger} = 0$. The constant in the BPY solution for the instrument in the two-period example (with the constraint binding in period 2) becomes the following:

$$\begin{aligned}
\gamma_{r,1} &= \left(Q + \zeta_1 \widetilde{H}_r^C\right)^{-1} \left(\left(\Theta_1^{-1}\widetilde{H}_r^C\right)' \beta V_{\tilde{x}\gamma,2} + \zeta_1 \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b\right) \\
&= \left(Q + \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)^{-1} \\
&\times \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b \quad \text{(F.65)}
\end{aligned}$$

which follows from:

$$\begin{aligned}
\zeta_1 &= \left(\Theta_1^{-1}\widetilde{H}_r^C\right)' \left(W + \beta V_{\tilde{x}\tilde{x},2}\right) \Theta_1^{-1} \\
&= \left(\Theta_1^{-1}\widetilde{H}_r^C\right)' W \Theta_1^{-1} \quad \text{(F.66)} \\
\Theta_1 &= \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x},2} + \widetilde{H}_r^F B_{r\tilde{x},2} \\
&= \widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B \quad \text{(F.67)}
\end{aligned}$$

and:

$$\begin{aligned}
B_{r\tilde{x},2} &= 0 \quad \text{(F.68)} \\
B_{\tilde{x}\tilde{x},2} &= -\Theta^{-1} \widetilde{H}_{\tilde{x}}^B \quad \text{(F.69)}
\end{aligned}$$

Application of the HP routine yields the following equivalent:

$$\begin{aligned}
\gamma_{r,1}^{\dagger} &= \Xi_{r\mu} \left(1 + \Delta_{\tilde{x}}\Xi_{\tilde{x}\mu}\right)^{-1} \Delta_r b \\
\gamma_{r,1}^{\dagger} &= \Xi_{r\mu} \Delta_r b \\
&= \left(Q + \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)^{-1} \\
&\times \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)^{-1} \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b \quad \text{(F.70)}
\end{aligned}$$

where the second line follows from $\Delta_{\tilde{x}} = 0$ (which is by virtue of the assumption that the model is either purely forward looking or that any lags in the system are not relevant for optimal policy) and:

$$\begin{aligned}
\Xi_{r\mu} &= \left(Q + \zeta \widetilde{H}_r^C\right)^{-1} \left(\left(\Theta^{-1}\widetilde{H}_r^C\right)' \beta V_{\tilde{x}\mu} - \zeta\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right)\right) \\
&= \left(Q + \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_{\tilde{x}}^B\right)\widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_{\tilde{x}}^B\right)\widetilde{H}_r^C\right)^{-1} \\
&\times \left(\left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_{\tilde{x}}^B\right)\widetilde{H}_r^C\right)' W \left(\widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_{\tilde{x}}^B\right)\left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_r^C - \widetilde{H}_r^F\right)\Delta_r^{-1}
\end{aligned} \quad \text{(F.71)}$$

which in turn follows from:

$$\begin{aligned}
\zeta &= \left(\Theta^{-1}\widetilde{H}_r^C\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right)\Theta^{-1} \\
&= \left(\Theta^{-1}\widetilde{H}_r^C\right)' W\Theta^{-1}
\end{aligned} \quad \text{(F.72)}$$

$$\begin{aligned}
\Theta &= \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x}} + \widetilde{H}_r^F B_{r\tilde{x}} \\
&= \widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1}\widetilde{H}_{\tilde{x}}^B
\end{aligned} \quad \text{(F.73)}$$

and:

$$B_{r\tilde{x}} = 0 \quad \text{(F.74)}$$

$$B_{\tilde{x}\tilde{x}} = -\Theta^{-1}\widetilde{H}_{\tilde{x}}^B \quad \text{(F.75)}$$

$$\Phi_{\tilde{x}\mu} = -\Theta^{-1}\widetilde{H}_r^C \Phi_{r\mu} \quad \text{(F.76)}$$

$$\Phi_{r\mu} = \Delta_r^{-1} \quad \text{(F.77)}$$

which is identical to the (correct) BPY solution.

This demonstrates that in the two-period example at least, application of the HP routine is valid for imposing instrument bound constraints under optimal discretionary policy when the model yields a static targeting condition. While the proof given here was not general, tests using forward-looking models for problems in which instrument constraints bind for multiple periods suggests that the proof could be generalized to multi-period settings. It is also worth noting that this special case in which the HP routine is valid is the same special case as that in which the standard rational expectations solution for the forward loadings on anticipated shocks is also valid – see Section 4.2 for a discussion.

### F.2.4 Why is application of the HP routine invalid away from this special case?

The reason why the HP routine is valid in the special case of models that yield a static targeting rule is that optimal decisions today are irrelevant for future losses. For the same reason, the value of the shadow shock necessary to impose the instrument bound constraint in period 2 is independent of period 1 optimal decisions. This suggests that the source of the failure of application of the HP routine in the general case is that it does not take account of the endogeneity of the future shadow shocks with respect to today's choices – the logic of the HP routine is based on the idea that shadow shocks are isomorphic to anticipated shocks and those are, by definition, assumed to be exogenous.

This reasoning can be formalized and tested. This section repeats the derivation in F.2.2, but proceeds on the basis that the policymaker accounts for the endogeneity of the period 2 shadow shocks with respect to period 1 decisions. In particular, application of the one-period only binding constraint case from Section F.1.2 implies that:

$$\mu_2 = -\Delta_{\tilde{x}}\widetilde{x}_1 + \Delta_r b \quad \text{(F.78)}$$

The endogeneity of $\mu_2$ can be embedded into the policymaker's optimal decisions by substituting out the shadow shocks in the rearranged model equations (F.39) from Section F.2.2 and then rearranging the result to get:[52]

$$\widetilde{x}_1 = -\kappa_1^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu}\right)\Delta_r b + \widetilde{H}_{\tilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^C r_1\right) \tag{F.79}$$

where:

$$\kappa_1 = \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F\left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right) + \widetilde{H}_r^F\left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right) \tag{F.80}$$

This can be embedded in the policymaker's loss minimisation problem as before:

$$\begin{aligned}&\min_{\widetilde{x}_1, r_1}\left(\widetilde{x}_1\right)' W\left(\widetilde{x}_1\right) + \left(r_1\right)' Q\left(r_1\right) + \beta\mathcal{L}_2 \\ &-2\lambda_1'\left(\widetilde{x}_1 + \kappa_1^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu}\right)\Delta_r b + \widetilde{H}_{\tilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^C r_1\right)\right)\end{aligned} \tag{F.81}$$

The first-order conditions are:

$$r_1: \quad 2\left(r_1\right)' Q + \beta\frac{\partial\mathcal{L}_2}{\partial r_1} - 2\lambda_1'\kappa_1^{-1}\widetilde{H}_r^C = 0 \tag{F.82}$$

$$\widetilde{x}_1: \quad 2\left(\widetilde{x}_1\right)' W + \beta\frac{\partial\mathcal{L}_2}{\partial\widetilde{x}_1} - 2\lambda_1' = 0 \tag{F.83}$$

$$\lambda_1: \quad \widetilde{x}_1 + \kappa_1^{-1}\left(\left(\widetilde{H}_{\tilde{x}}^F\Phi_{\tilde{x}\mu} + \widetilde{H}_r^F\Phi_{r\mu}\right)\Delta_r b + \widetilde{H}_{\tilde{x}}^B\widetilde{x}_0 + \widetilde{H}_r^C r_1\right) = 0 \tag{F.84}$$

The recognition that the shadow shocks are endogenous also has implications for the marginal effect of current decisions on future losses. Losses in period 2 can be written as:

$$\mathcal{L}_2 = \left(\widetilde{x}_2\right)' W\left(\widetilde{x}_2\right) + \left(r_2\right)' Q\left(r_2\right) + \beta\mathcal{L}_3 \tag{F.85}$$

Given that the instrument is unconstrained from period 3, it is the case that:

$$\mathcal{L}_3 = \left(\widetilde{x}_2\right)' V_{\tilde{x}\tilde{x}}\left(\widetilde{x}_2\right) \tag{F.86}$$

Substituting this into the expression for $\mathcal{L}_2$ gives:

$$\mathcal{L}_2 = \left(\widetilde{x}_2\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\left(\widetilde{x}_2\right) + \left(r_2\right)' Q\left(r_2\right) \tag{F.87}$$

and using the period 2 laws of motion for $\widetilde{x}_2$ and $r_2$, having substituted out $\mu_2$ and rearranging gives:[53]

$$\mathcal{L}_2 = \left(\widetilde{x}_1\right)' V_{\tilde{x}\tilde{x},2}^*\widetilde{x}_1 + \left(\widetilde{x}_1\right)' V_{\tilde{x}\mu,2}^* + \left(V_{\tilde{x}\mu,2}^*\right)'\widetilde{x}_1 + \mathcal{C} \tag{F.88}$$

where:

$$V_{\tilde{x}\tilde{x},2}^* = \left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right) + \left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right)' Q\left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right) \tag{F.89}$$

$$V_{\tilde{x}\mu,2}^* = \left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right)'\left(W + \beta V_{\tilde{x}\tilde{x}}\right)\Phi_{\tilde{x}\mu}\Delta_r b + \left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right)' Q\Phi_{r\mu}\Delta_r b \tag{F.90}$$

and $\mathcal{C}$ is a constant that is a function of $b$ and parameters only. It is straightforward to see that $\frac{\partial\mathcal{L}_2}{\partial r_1} = 0$. It is also straightforward to see that:

$$\frac{\partial\mathcal{L}_2}{\partial\widetilde{x}_1} = 2\left(\widetilde{x}_1\right)' V_{\tilde{x}\tilde{x},2}^* + 2\left(V_{\tilde{x}\mu,2}^*\right)' \tag{F.91}$$

[52]Note that $\kappa$ has a time subscript reflecting that it would, in general, depend on the number (and precise sequence) of future periods in which the instrument bound constraint is expected to bind.

[53]Again, $V_{\tilde{x}\tilde{x},1}^*$ and $V_{\tilde{x}\mu,1}^*$ have time subscripts in recognition of the fact that these coefficients depend on the number (and sequence) of future periods in which the instrument bound constraints are expected to bind.

This expression can be substituted into the FOC for the endogenous variables and the result rearranged to give an expression for $\lambda_1'$:

$$\lambda_1' = (\widetilde{x}_1)' \left(W + \beta V^*_{\tilde{x}\tilde{x},2}\right) + (\mu_2)' \beta \left(V^*_{\tilde{x}\mu,2}\right)' \tag{F.92}$$

This in turn can be substituted into the FOC for the instrument to get:

$$\begin{aligned} (r_1)' Q - \left((\widetilde{x}_1)' \left(W + \beta V^*_{\tilde{x}\tilde{x},2}\right) + \beta \left(V^*_{\tilde{x}\mu,2}\right)'\right) \kappa_1^{-1} \widetilde{H}_r^C &= 0 \\ Q r_1 - \left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \left(W + \beta V^*_{\tilde{x}\tilde{x},2}\right) \widetilde{x}_1 - \left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \beta V^*_{\tilde{x}\mu,2} &= 0 \end{aligned} \tag{F.93}$$

Similarly, $\widetilde{x}_1$ can be eliminated using the FOC for the Lagrange multiplier to get:

$$Q r_1 + \chi_1 \left(\left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right) \Delta_r b + \widetilde{H}_{\tilde{x}}^B \widetilde{x}_0 + \widetilde{H}_r^C r_1\right) - \left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \beta V^*_{\tilde{x}\mu,2} = 0 \tag{F.94}$$

where:

$$\chi_1 = \left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \left(W + \beta V^*_{\tilde{x}\tilde{x},2}\right) \kappa_1^{-1} \tag{F.95}$$

which can be rearranged and written as:

$$r_1 = B^*_{r,\tilde{x},1} \widetilde{x}_0 + \gamma^*_{r,1} \tag{F.96}$$

where:

$$B^*_{r,\tilde{x},1} = -\left(Q + \chi_1 \widetilde{H}_r^C\right)^{-1} \chi_1 \widetilde{H}_{\tilde{x}}^B \tag{F.97}$$

$$\begin{aligned} \gamma^*_{r,1} &= \left(Q + \chi_1 \widetilde{H}_r^C\right)^{-1} \left(\left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \beta V^*_{\tilde{x}\mu,2} - \chi_1 \left(\widetilde{H}_{\tilde{x}}^F \Phi_{\tilde{x}\mu} + \widetilde{H}_r^F \Phi_{r\mu}\right) \Delta_r b\right) \\ &= \left(Q + \chi_1 \widetilde{H}_r^C\right)^{-1} \left(\left(\kappa_1^{-1} \widetilde{H}_r^C\right)' \beta V^*_{\tilde{x}\mu,2} + \chi_1 \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b\right) \end{aligned} \tag{F.98}$$

where the third line follows from substituting in the known expressions for $\Phi_{\tilde{x}\mu}$ and $\Phi_{r\mu}$ from Section F.1.2.

The equivalent BPY expressions copied from Section F.2.1 are:

$$B_{r\tilde{x},1} = -\left(Q + \zeta_1 \widetilde{H}_r^C\right)^{-1} \zeta_1 \widetilde{H}_{\tilde{x}}^B \tag{F.99}$$

$$\gamma_{r,1} = \left(Q + \zeta_1 \widetilde{H}_r^C\right)^{-1} \left(\left(\Theta_1^{-1} \widetilde{H}_r^C\right)' \beta V_{\tilde{x}\gamma,2} + \zeta_1 \left(\widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_r^C - \widetilde{H}_r^F\right) b\right) \tag{F.100}$$

It is clear that these expressions are identical if: $V^*_{\tilde{x}\mu,2} = V_{\tilde{x}\gamma,2}$, $\chi_1 = \zeta_1$ and $\kappa_1 = \Theta_1$. The (correct) BPY expressions from Section F.2.1 are:

$$V_{\tilde{x}\gamma,2} = \left(\Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Theta^{-1} \widetilde{H}_r^C b \tag{F.101}$$

$$\zeta_1 = \left(\Theta_1^{-1} \widetilde{H}_r^C\right)' \left(W + \beta V_{\tilde{x}\tilde{x},2}\right) \Theta_1^{-1} \tag{F.102}$$

$$\begin{aligned} \Theta_1 &= \widetilde{H}_{\tilde{x}}^C + \widetilde{H}_{\tilde{x}}^F B_{\tilde{x}\tilde{x},2} + \widetilde{H}_r^F B_{r\tilde{x},2} \\ &= \widetilde{H}_{\tilde{x}}^C - \widetilde{H}_{\tilde{x}}^F \Theta^{-1} \widetilde{H}_{\tilde{x}}^B \end{aligned} \tag{F.103}$$

where the final line substitutes in $B_{r\tilde{x},2} = 0$ and $B_{\tilde{x}\tilde{x},2} = -\Theta^{-1} \widetilde{H}_{\tilde{x}}^B$ and where:

$$V_{\tilde{x}\tilde{x},2} = \left(\Theta^{-1} \widetilde{H}_{\tilde{x}}^B\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Theta^{-1} \widetilde{H}_{\tilde{x}}^B \tag{F.104}$$

Starting with $V^*_{\tilde{x}\mu,2}$, application of a variant of the HP method extended to recognize the endogeneity of $\mu$ gives:

$$\begin{aligned}
V^*_{\tilde{x}\mu,2} &= \left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Phi_{\tilde{x}\mu}\Delta_r b + \left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right)' Q \Phi_{r\mu}\Delta_r b \\
&= \left(\Theta^{-1}\left(\widetilde{H}^B_{\tilde{x}} + \widetilde{H}^C_r \Delta_r^{-1}\Delta_{\tilde{x}}\right) - \Theta^{-1}\widetilde{H}^C_r \Delta_r^{-1}\Delta_{\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Theta^{-1}\widetilde{H}^C_r \Delta_r^{-1}\Delta_r b \\
&+ \left(\Delta_r^{-1}\Delta_{\tilde{x}} - \Delta_r^{-1}\Delta_{\tilde{x}}\right)' Q \Delta_r^{-1}\Delta_r b \\
&= \left(\Theta^{-1}\widetilde{H}^B_{\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \Theta^{-1}\widetilde{H}^C_r b
\end{aligned} \tag{F.105}$$

where the second and third lines substitute in the expressions for $\Phi_{\tilde{x}\mu}$, $\Phi_{r\mu}$, $B_{\tilde{x}\tilde{x}}$ and $B_{r\tilde{x}}$. This expression is identical to the BPY equivalent. A necessary condition for $\chi_1 = \zeta_1$ is that $V^*_{\tilde{x}\tilde{x},2} = V^*_{\tilde{x}\tilde{x}}$.

First, $V^*_{\tilde{x}\tilde{x},2}$ can be written as:

$$\begin{aligned}
V^*_{\tilde{x}\tilde{x},2} &= \left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right) + \left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right)' Q \left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right) \\
&= \left(\Theta^{-1}\widetilde{H}^B_{\tilde{x}}\right)' \left(W + \beta V_{\tilde{x}\tilde{x}}\right) \left(\Theta^{-1}\widetilde{H}^B_{\tilde{x}}\right)'
\end{aligned} \tag{F.106}$$

which follows from the same substitutions as into $V^*_{\tilde{x}\mu,2}$ and is also identical to the BPY expression.

Turning to $\chi_1$ and $\kappa_1$, the 'endogenous HP' expression for $\chi_1$ is:

$$\chi_1 = \left(\kappa_1^{-1}\widetilde{H}^C_r\right)' \left(W + \beta V^*_{\tilde{x}\tilde{x},2}\right) \kappa_1^{-1} \tag{F.107}$$

Since it has already been established that $V^*_{\tilde{x}\tilde{x},2} = V^*_{\tilde{x}\tilde{x}}$, $\chi_1$ is identical to $\zeta_1$ if $\kappa_1 = \Theta_1$. The expression for $\kappa_1$ can be written as:

$$\begin{aligned}
\kappa_1 &= \widetilde{H}^C_{\tilde{x}} + \widetilde{H}^F_{\tilde{x}}\left(B_{\tilde{x}\tilde{x}} - \Phi_{\tilde{x}\mu}\Delta_{\tilde{x}}\right) + \widetilde{H}^F_r\left(B_{r\tilde{x}} - \Phi_{r\mu}\Delta_{\tilde{x}}\right) \\
&= \widetilde{H}^C_{\tilde{x}} - \widetilde{H}^F_{\tilde{x}}\Theta^{-1}\widetilde{H}^B_{\tilde{x}}
\end{aligned} \tag{F.108}$$

which is identical to the BPY expression for $\Theta_1$, thereby proving, in this two-period example at least, that the source of the failure of application of the HP algorithm for imposing bound constraints in the case of optimal discretionary policy is a failure to take into account the endogeneity of future shadow shocks with respect to current decisions. Intuitively, a policymaker can seek to mitigate the impact of a bound constraint on the discounted sum of future losses in the optimal decisions they make. In treating the shadow shocks as exogenous (and isomorphic to anticipated shocks), an unadjusted application of the HP routine fails to take that into account.

## F.3 Summary

Do the preceding investigations imply that an 'endogenous HP' routine (that corrects for the dependence of future $\mu$ on current and future states) could be used in place of BPY to more efficiently compute solutions in which instrument bound constraints may occasionally bind? Unfortunately, the answer is 'no'. In making the endogenous correction to the HP algorithm, in line with the particular example studied above, it was assumed that *it was known* that the constraint was binding in period 2, but not in any other periods. Thus, to apply a correction to the HP algorithm in a case in which the periods in which any constraints are binding are unknown would require a conjecture for the periods in which the constraints bind. As a result, such an approach would require the same 'guess and verify' inefficiency inherent in the BPY approach.

# G Optimal commitment solution with non-policy OBCs

The solution is analogous to those for optimal discretion with anticipated disturbances and follows a backward induction approach. The starting point is the structural form of the model given in equation (83), and repeated here for convenience:

$$H_y^F y_{t+1} + H_{y,t}^C y_t + H_y^B y_{t-1} = \widehat{\Psi}_{\hat{z}} \widehat{z}_t + \Psi_\mu \mu_t \tag{G.1}$$

## G.1 Period $H$

The assumption that the model is in the 'baseline' state from period $H+1$ onwards implies that:

$$y_{H+1} = B_y y_H$$

where $B_y$ is the rational expectations solution matrix derived in Section 3.

Using this result in the structural form of the model for date $H$ implies that:

$$y_H = B_{y,H} y_{H-1} + \widehat{\Phi}_H \widehat{z}_H + \Phi_{\mu,H} \mu_H$$

where

$$\begin{aligned} B_{y,H} &= -\left(H_y^F B_{y,H+1} + H_{y,H}^C\right)^{-1} H_y^B \\ \widehat{\Phi}_H &= \left(H_y^F B_{y,H+1} + H_{y,H}^C\right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,H} &= \left(H_y^F B_{y,H+1} + H_{y,H}^C\right)^{-1} \Psi_\mu \end{aligned}$$

## G.2 Period $H-1$

In period $H-1$, the structural form of the model is:

$$H_y^F y_H + H_{y,H-1}^C y_{H-1} + H_y^B y_{H-2} = \widehat{\Psi}_{\hat{z}} \widehat{z}_{H-1} + \Psi_\mu \mu_{H-1}$$

Using the solution from period $H$ gives:

$$H_y^F \left(B_{y,H} y_{H-1} + \widehat{\Phi}_H \widehat{z}_H + \Phi_{\mu,H} \mu_H\right) + H_{y,H-1}^C y_{H-1} + H_y^B \widehat{y}_{H-2} = \widehat{\Psi}_{\hat{z}} \widehat{z}_{H-1} + \Psi_\mu \mu_{H-1}$$

which implies that the solution for $y_{H-1}$ satisfies:

$$y_{H-1} = B_{y,T-1} y_{H-2} + \widehat{\Phi}_{H-1} \widehat{z}_{H-1} + F_{H-1,1} \widehat{\Phi}_H \widehat{z}_H + \Phi_{\mu,H-1} \mu_{H-1} + F_{H-1,1} \Phi_{\mu,H} \mu_H$$

where

$$\begin{aligned} B_{y,H-1} &= -\left(H_y^F B_{y,H} + H_{y,H-1}^C\right)^{-1} H_y^B \\ \widehat{\Phi}_{H-1} &= \left(H_y^F B_{y,H} + H_{y,H-1}^C\right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,H-1} &= \left(H_y^F B_{y,H} + H_{y,H-1}^C\right)^{-1} \Psi_\mu \\ F_{H-1,1} &= -\left(H_y^F B_{y,H} + H_{y,H-1}^C\right)^{-1} H_y^F \end{aligned}$$

## G.3 Period $H-2$

The structural form is:

$$H_y^F y_{H-1} + H_{y,H-2}^C y_{H-2} + H_y^B y_{H-3} = \widehat{\Psi}_{\hat{z}} \widehat{z}_{H-2} + \Psi_\mu \mu_{H-2}$$

and plugging in the solution for $y_{H-1}$ gives:

$$\begin{aligned} & H_y^F \left( B_{y,H-1} y_{H-2} + \widehat{\Phi}_{H-1} \widehat{z}_{H-1} + F_{H-1,1} \widehat{\Phi}_H \widehat{z}_H + \Phi_{\mu,H-1} \mu_{H-1} + F_{H-1,1} \Phi_{\mu,H} \mu_H \right) \\ & + H_{y,H-2}^C y_{H-2} + H_y^B y_{H-3} = \widehat{\Psi}_{\hat{z}} \widehat{z}_{H-2} + \Psi_\mu \mu_{H-2} \end{aligned}$$

so that:

$$\begin{aligned} y_{H-2} = & \; B_{y,H-2} y_{H-3} + \widehat{\Phi}_{H-2} \widehat{z}_{H-2} + F_{H-2,1} \widehat{\Phi}_{H-1} \widehat{z}_{H-1} + F_{H-2,2} \widehat{\Phi}_H \widehat{z}_H \\ & + \Phi_{\mu,H-2} \mu_{H-2} + F_{H-2,1} \Phi_{\mu,H-1} \mu_{H-1} + F_{H-2,2} \Phi_{\mu,H} \mu_H \end{aligned}$$

where

$$\begin{aligned} B_{y,H-2} &= -\left( H_y^F B_{y,H-1} + H_{y,H-2}^C \right)^{-1} H_y^B \\ \widehat{\Phi}_{H-2} &= \left( H_y^F B_{y,H-1} + H_{y,H-2}^C \right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,H-2} &= \left( H_y^F B_{y,H-1} + H_{y,H-2}^C \right)^{-1} \Psi_\mu \\ F_{H-2,1} &= -\left( H_y^F B_{y,H-1} + H_{y,H-2}^C \right)^{-1} H_y^F \\ F_{H-2,2} &= -\left( H_y^F B_{y,H-1} + H_{y,H-2}^C \right)^{-1} H_y^F F_{H-1,1} \end{aligned}$$

## G.4 Period $H-3$

Using the $H-2$ solution in the $H-3$ structural form implies that:

$$\begin{aligned} & H_y^F \begin{pmatrix} B_{y,H-2} y_{H-3} + \widehat{\Phi}_{H-2} \widehat{z}_{H-2} + F_{H-2,1} \widehat{\Phi}_{H-1} \widehat{z}_{H-1} + F_{H-2,2} \widehat{\Phi}_H \widehat{z}_H \\ + \Phi_{\mu,H-2} \mu_{H-2} + F_{H-2,1} \Phi_{\mu,H-1} \mu_{H-1} + F_{H-2,2} \Phi_{\mu,H} \mu_H \end{pmatrix} \\ & + H_{y,H-3}^C y_{H-3} + H_y^B y_{H-4} = \widehat{\Psi}_{\hat{z}} \widehat{z}_{H-3} + \Psi_\mu \mu_{H-3} \end{aligned}$$

so that:

$$\begin{aligned} y_{H-3} = & \; B_{y,H-3} y_{H-4} + \widehat{\Phi}_{H-3} \widehat{z}_{H-3} + F_{H-3,1} \widehat{\Phi}_{H-2} \widehat{z}_{H-2} + F_{H-3,2} \widehat{\Phi}_{H-1} \widehat{z}_{H-1} + F_{H-3,3} \widehat{\Phi}_H \widehat{z}_H \\ & + \; \Phi_{\mu,H-3} \mu_{H-3} + F_{H-3,1} \Phi_{\mu,H-2} \mu_{H-2} + F_{H-3,2} \Phi_{\mu,H-1} \mu_{H-1} + F_{H-3,3} \Phi_{\mu,H} \mu_H \end{aligned}$$

where

$$\begin{aligned} B_{H-3} &= -\left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} H_y^B \\ \widehat{\Phi}_{H-3} &= \left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,H-3} &= \left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} \Psi_\mu \\ F_{H-3,1} &= -\left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} H_y^F \\ F_{H-3,2} &= -\left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} H_y^F F_{H-2,1} \\ F_{H-3,3} &= -\left( H_y^F B_{y,H-2} + H_{y,H-3}^C \right)^{-1} H_y^F F_{H-2,2} \end{aligned}$$

## G.5 Period $t$

The preceding recursions reveal that the the general solution in period $t$ is given by:

$$y_t = B_{y,t}y_{t-1} + \sum_{h=0}^{H-t} F_{t,h}\widehat{\Phi}_{t+h}\widehat{z}_{t+h} + \sum_{h=0}^{H-t} F_{t,h}\Phi_{\mu,t+h}\mu_{t+h}$$

where

$$\begin{aligned} B_{y,t} &= -\left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} H_y^B \\ \widehat{\Phi}_t &= \left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} \widehat{\Psi}_{\hat{z}} \\ \Phi_{\mu,t} &= \left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} \Psi_\mu \end{aligned}$$

and

$$B_{y,H+1} = B_y$$

where $B_y$ is the rational expectations solution from equation (14).

For $h > 1$, the $F$ matrices are given by:

$$F_{t,h} = \Upsilon_t F_{t+1,h-1}$$

with

$$F_{t,0} = \mathbb{I}$$

which implies that $F_{t,1} = \Upsilon_t$, where

$$\Upsilon_t \equiv -\left(H_y^F B_{y,t+1} + H_{y,t}^C\right)^{-1} H_y^F$$